1-1-1932

A problem related to an oblique cone

James Milton Reynolds
Atlanta University

THE ATLANTA UNIVERSITY

A PROBLEM RELATED TO AN OBLIQUE CONE

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

IN CANDIDACY FOR

THE DEGREE OF MASTER OF ARTS

DEPARTMENT OF MATHEMATICS

BY

JAMES MILTON REYNOLDS

ATLANTA GEORGIA

JUNE 1932

UMI Number: EP17444

UMI®

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

CONTENTS

Illustrations

CHAPTER I

INTRODUCTION

The Problem

Given an oblique cone whose base is a circle of radius r and vertex at V. Required the lateral area of the cone.

First, we shall set up the required area in terms of a single variable θ, using a single integral. A general formula will be worked out and the following five special cases considered.

(a) When altitude falls at center of base, and altitude is not 0.

(b) When altitude falls at center of base, and altitude is 0.

(c) When altitude falls inside of base, and altitude is 0.

(d) When altitude falls outside of base, and altitude is 0.

(e) When altitude falls on circumference of circle, and altitude is 0.

The integral involved is elliptic and will be reduced to an integrable form by an elaborate process of transformations. The general formula thus obtained contains all three types of the standard elliptic integrals. The values of these for specific cases may be obtained by reference to tables.

<u>Auxiliary Construction and Notations</u>. -

1. Let a and b be elements of the cone from the extremities of the same diameter, AB, of the base, with b less than a, b drawn from A.

2. VD (=1), the altitude of the cone, is perpendicular to DQ, a line in the plane of the base, parallel to O P_2.
3. AD = A'D, where A is on the base of the cone and A' is outside of the base.
4. P_1 and P_2 are two neighboring points on the base of the cone so that the chord P_1P_2 subtends an angle $\Delta\theta$ at the center of the circle. The $\lim_{\Delta\theta \to 0} \Delta P_1P_2V$ is an element of the area of the cone.
5. O is the center of the base of the cone, OD = m.
6. Let chord P_1P_2 extended intersect DQ at M. Draw a line through O parallel to P_1P_2 intersecting DQ at R. RM = r as P_1 approaches P_2.
7. Triangle AVA' is isosceles.
8. VM is altitude of P_1P_2V; i. e., VM is perpendicular to P_1P_2M.
9. OP_2 makes an angle θ with OA; hence, θ is a variable since P_2 is any point on the base.
10. S = lateral area.

<u>Fundamental Equations.</u> - Observing figure 1, by construction angle VDM = rt. angle. M R O is a rt. angle in limit.

$VM^2 = MD^2 + 1^2 = (DR - r)^2 + 1^2 = (m\cos\theta - r)^2 + 1^2.$

From Plane Geometry

area triangle $P_1P_2V = 1/2\ r\Delta\theta\sqrt{(m\cos\theta - r)^2 + 1^2}$.

Therefore, $S/2 = \lim_{\Delta\theta \to 0} \sum_{\theta=0}^{\theta=\pi} 1/2\ r\Delta\theta\sqrt{(m\cos\theta - r)^2 + 1^2}$,

$$S/2 = 1/2\int_0^{\pi} r\sqrt{(m\cos\theta - r)^2 + 1^2}\ d\theta$$

and

$$S = r\int_0^{\pi}\sqrt{(m\cos\theta - r)^2 + 1^2}\ d\theta\ . \qquad (\alpha)$$

CHAPTER II

GENERAL SOLUTION OF THE EQUATION

Reduction to Integral of an Odd and Even Function

<u>Trigonometric Transformation</u>. - Our first step in reducing (α) is to put $\cos\theta = x$.

Then $dx = -\sin\theta\, d\theta$, and $d\theta = -\dfrac{dx}{\sqrt{1 - x^2}}$,

when $\theta = \pi$, $x = -1$, $\theta = 0$, $x = +1$.

Substitution in the integrand gives

$$S = r\int_{-1}^{1}\sqrt{\frac{(mx - r)^2 + l^2}{(1 - x)^2}}\; dx\,, \qquad (\alpha')$$

or

$$S = r\int_{-1}^{1}\frac{\left(x^2 - \frac{2\,rx}{m} + \frac{r^2 + l^2}{m^2}\right)\; dx}{\sqrt{-(x^2 - 1)}\,\sqrt{x^2 - \frac{2\,rx}{m} + \frac{r^2 + l^2}{m^2}}}\,,$$

which may be written

$$S = r\int_{-1}^{1}\frac{(x^2 - 2gx + h)\;\; dx}{\sqrt{x^2 - 2g'x + h'}\,\sqrt{x^2 - 2gx + h}}\,, \qquad (\beta)$$

where $g' = 0$, $h' = -1$, $g = \frac{r}{m}$, $h = \frac{r^2 + l^2}{m^2}$.

<u>Linear Fractional Transformation</u>. - Here we shall make a linear fractional transformation,

$x = \frac{p + qy}{1 + y}$, where p and q are parameters to be so chosen that there will be no first degree term in y after the transformation, and p and q must be real.

$$\text{Since} \quad x = \frac{p + qy}{1 + y} \qquad \text{(a)}$$

$$dx = \frac{q - p}{(1 + y)^2}\,dy \qquad \text{(differentiating)}$$

Also by (a),

$$y = \frac{p - 1}{1 - q} \quad \text{when } x = 1, \text{ and } y = -\frac{p + 1}{q + 1}$$

when $x = -1$. (b)

Substituting (a) and (b) in (B)

$$S = mr\int_{-\frac{p+1}{q+1}}^{\frac{p-1}{1-q}} \frac{\left[\frac{(p + qy)^2}{(1 + y)^2} - 2g\left(\frac{p + qy}{1 + y}\right) + h\right]\left(\frac{q - p}{(1 + y)^2}\right)dy}{\sqrt{-\left[\frac{(p + qy)^2}{(1 + y)^2} + h'\right]}\sqrt{\left[\frac{(p + qy)^2}{(1 + y)^2} - 2q\left(\frac{p + qy}{1 + y}\right) + h\right]}}. \qquad \text{(c)}$$

Performing indicated operation in (c), we have

$$S = mr\int_{-\frac{p+1}{q+1}}^{\frac{p-1}{1-q}} \frac{(A + By + Cy^2)\,dy\cdot(q-p)}{(1 + y)^2\sqrt{A' + B'y + C'y^2}\,\sqrt{A + By + Cy^2}}; \qquad \text{(d)}$$

Where

$$\left.\begin{array}{ll} A = p^2 - 2gp + h, & A' = p^2 + h', \\ B = 2pq - 2g(q + p) + 2h, & B' = 2(pq + h'), \\ C = q^2 - 2gq + h, & C' = q^2 + h'. \end{array}\right\}. \qquad \text{(e)}$$

<u>Parametric Representation of p + q.</u> - The known parameters in this problem are a, b, m, r, l. In this section p and q will be determined in terms of a and b. In ensuing sections, certain other constant terms will also be represented in terms of a and b for convenience in calculations. Setting B and B' = o in (e), and solving for p and q, we have

$$2pq - g(q + p) + 2h = o$$

$$2pq + 2h' = o$$

$$q + p = \frac{h - h'}{g}$$

$$pq = -h'.$$

By section on trigonometric transformation,

$$q + p = \frac{h - h'}{g} = \frac{m^2 + r^2 + 1^2}{rm} . \qquad (1)$$

Also,

$$q = \frac{1}{p} , \quad p = \frac{1}{q} .$$

Substituting in (1),

$$p + \frac{1}{p} = \frac{m^2 + r^2 + 1^2}{rm} .$$

We form an equation quadratic in p:

$$rmp^2 - (m^2 + r^2 + 1^2)\, p + rm = 0$$

the solution of which gives (2)

$$p = \frac{(m^2 + r^2 + 1^2) \pm \sqrt{(m^2 + r^2 + 1^2)^2 - 4r^2m^2}}{2rm} .$$

The expression under the radical is recognized as the product of the sum and difference of the two numbers $(m^2 + r^2 + 1^2)$ and 2rm. Observing figure one, we note

$$a^2 = r^2 + m^2 + 1^2 + 2\, rm ,$$
$$b^2 = r^2 + m^2 + 1^2 - 2\, rm,$$

adding, $\frac{a^2 + b^2}{2} = r^2 + m^2 + 1^2,$

subtracting, $\frac{a^2 - b^2}{2} = 2rm.$

Operating on (2),

$$p = \frac{\frac{a^2 + b^2}{2} \pm \sqrt{a^2 \cdot b^2}}{\frac{a^2 - b^2}{2}} ,$$

$$p = \frac{a - b}{a + b} , \quad \frac{a + b}{a - b},$$

$$q = \frac{1}{p} = \frac{a + b}{a - b} , \quad \frac{a - b}{a + b} .$$

We choose $p = \frac{a - b}{a + b} , \quad q = \frac{a + b}{a - b} .$

The limits in terms of a and b are:

$$\frac{p-1}{1-q} = \frac{a-b}{a+b}, (=p)$$

$$-\frac{p+1}{q+1} = -\frac{a-b}{a+b}. (=-p) .$$

Also, q - p in terms of a and b becomes $\frac{4ab}{a^2-b^2}$.

Making use of these values, we get the transformed equation.

<u>The Transformed Equation</u>. -

$$S = mr\int_{-p}^{p} \frac{(A+Cy^2)\frac{4ab}{a^2-b^2}dy}{(1+y^2)\sqrt{-(A'+C'y^2)(A+Cy^2)}} . \qquad (\gamma)$$

In (γ), allow A' to absorb the - sign and divide the first binomial of the radical in the denominator by $-A'$; also divide the second binomial of the radical and the one of the numerator by A. Then

$$S = mr\int_{-p}^{p} \frac{\frac{4ab}{a^2-b^2} A\left(1+\frac{C}{A}y^2\right)dy}{(1+y^2)\sqrt{-A'A\left(1-\frac{C'}{A'}y^2\right)\left(1+\frac{C}{A}y^2\right)}} . \qquad (a)$$

In (a) put

$$\rho^2 = \frac{C}{A} = \frac{q^2-2gq+h}{p^2-2gq+h}, \qquad (b)$$

$$\sigma^2 = \frac{C'}{A'} = \frac{q^2+h'}{h'-p^2} .$$

By section on parametric representation of p and q,

$$2\,rm = \frac{a^2-b^2}{2},$$

$$r = \frac{a^2-b^2}{4m} .$$

Likewise, since $m^2+r^2+1^2 = \frac{a^2+b^2}{2}$,

$$h = \frac{a^2+b^2-2m^2}{2m^2} = \frac{a^2+b^2}{2m^2} - 1 .$$

Write (a)

$$S = mr \int_{-p}^{p} \frac{\frac{4ab}{a^2 - b^2} \frac{A}{\sqrt{-A'A}} (1 + \rho^2 y^2) \, dy}{(1 + y)^2 \sqrt{(1 + \rho^2 y^2)(1 - \sigma^2 y^2)}} . \qquad (c)$$

Let us define our constant terms by the parameters a, b, r, m. Substituting in (b):

$$\rho^2 = \frac{\frac{(a+b)^2}{(a+b)^2} - \frac{2}{m} \frac{(a^2 - b^2)}{4m} \frac{(a+b)}{(a-b)} + \frac{a^2 + b^2 - 2m^2}{2m^2}}{\frac{(a-b)^2}{(a+b)^2} - \frac{2}{m} \frac{(a^2 - b^2)}{4m} \frac{(a-b)}{(a+b)} + \frac{a^2 + b^2 - 2m^2}{2m^2}} . \qquad (d)$$

Multiplying both numerator and denominator of (d) by $(a + b)^2 (a - b)^2$ we get an expression which reduces to

$$\rho^2 = \frac{(a + b)^2 - 4r^2}{4r^2 - (a - b)^2} .$$

Let $r^2 = \left[\frac{a^2 - b^2}{4m}\right]^2$ and multiply numerator of (d) by $(a - b)^2$ and denominator by $(a + b)^2$.

$$\rho^2 = \frac{(a + b)^2}{(a - b)^2} \left[\frac{4 m^2 - (a - b)^2}{(a + b)^2 - 4 m^2} \right] ;$$

similarly,

$$\sigma^2 = \frac{\frac{(a + b)^2}{(a - b)^2} - 1}{1 - \frac{(a - b)^2}{(a + b)^2}} = \frac{(a + b)^2}{(a - b)^2} ;$$

and

$$A = \left[\frac{(a - b)^2}{(a + b)^2} - \frac{2r}{m} \frac{(a - b)}{(a + b)} + \frac{a^2 + b^2 - 2m^2}{2m^2} \right]$$ which

reduces to

$$A = \frac{4 ab}{(a + b)^2 (a - b)} [(4r^2 - (a - b)^2] .$$

Then

$$\sqrt{-A'A} = \sqrt{\left[1 - \frac{(a - b)^2}{(a + b)^2} \right] \left[\frac{4 ab}{(a + b)^2 (a - b)} \right] [4r^2 - (a-b)^2]}$$

$$= \frac{4 ab}{(a + b)^2 (a - b)} \sqrt{4 r^2 - (a - b)^2} .$$

Therefore, our constant term of (c) ,

$$\frac{4\ ab\ mr\ A}{\sqrt{-A'A}\ (a^2 - b^2)^2}\text{, becomes, defining it as M.}$$

$$M = \frac{\dfrac{1}{a^2 - b^2} \cdot 4\ ab \cdot \dfrac{4\ ab\left[4r^2 - (a-b)^2\right]}{(a+b)^2\ (a-b)^2} \cdot m \cdot \dfrac{(a^2 - b^2)}{4m}}{\dfrac{4\ ab}{(a+b)^2\ (a-b)}\sqrt{4r^2 - (a-b)^2}} .$$

Hence, $M = \dfrac{ab}{a - b}\sqrt{4\ r^2 - (a - b)^2}$.

Let $\sqrt{Y} = \sqrt{(1 + \rho^2 y^2)\ (1 - \sigma^2 y^2)}$.

Using the above values in (c):

$$S = M \int_{p}^{p} \frac{(1 + \rho^2 y^2)\ dy}{(1 + y)^2\ \sqrt{Y}} . \qquad (d)$$

<u>The Rational Function</u> . - Rationalizing (d) of the last section by multiplying both numerator and denominator by $(1 - y)^2$ we have

$$M \int_{-p}^{p} \frac{(1 + \rho^2 y^2)\ dy}{(1 + y)^2\ \sqrt{Y}} \cdot \frac{(1 - y)^2}{(1 - y)^2} = M \int_{-p}^{p} \frac{[1 - 2y + (1 + \rho^2) y^2 - 2\rho^2 y^3 + \rho^2 y^4]\ dy}{[1 - 2y^2 + y^4]\ \sqrt{Y}}$$

$$= M \int_{-p}^{p} \frac{[1 + (1 + \rho^2)\ y^2 + \rho^2 y^4]\ dy}{(1 - y^2)^2\ \sqrt{Y}} - M \int_{-p}^{p} \frac{(2y + 2\rho^2 y^3)\ dy}{(1 - y^2)^2\ \sqrt{Y}} \qquad (\delta)$$

From Odd and Even Fcns To Standard Form

<u>The Odd Fcn.</u> - Consider the second member of (δ):

$$M \int_{-p}^{p} \frac{(2y + 2\ \rho^2 y^3)\ dy}{(1 - y^2)^2\ \sqrt{Y}} .$$

Obviously this is an odd fcn. and since its limits are from - p to + p, its value is 0. We now need only consider the first member in determining S.

<u>The Even Fcn.</u> - <u>Partial Fractions</u>. - Integrating

$S = M \int_{p}^{p} \dfrac{[1 + (1 + \rho^2)\ y^2 + \rho^2 y^4]}{(1 - y^2)^2\ \sqrt{Y}}\ dy$ by partial fractions

$$S = M \int_{-p}^{p} \frac{\rho^2}{\sqrt{Y}}\, dy + M \int_{p}^{p} \frac{A\, dy}{(1-y^2)^2 \sqrt{Y}} + M \int_{p}^{p} \frac{B\, dy}{(1-y^2)\sqrt{Y}} \qquad (\varepsilon)$$

where $A = 2\,(1 + \rho^2)$ $B = -\,(1 + 3\rho^2)$.

We shall treat each integral separately, calling them for convenience the ρ^2 integral, the A - integral and the B - integral. Each of these integrals reduces to standard form by putting $y^2 = \frac{1 - t^2}{\sigma^2}$. This transformation was taken from a table of transformations prepared in the course in Elliptic Integrals. The combination of signs in $\sqrt{Y}$ determines the transformation to be used. Operating on $y^2 = \frac{1 - t^2}{\sigma^2}$ we have

$$y = \sqrt{\frac{1 - t^2}{\sigma^2}}, \qquad dy = -\frac{t\, dt}{\sigma\sqrt{1 - t^2}}, \qquad t = \sqrt{1 - \sigma^2 y^2}.$$

The limits become, using the value of t, t = 1, t = 0.

$$\sqrt{Y} = \sqrt{(1 + \rho^2 y^2)(1 - \sigma^2 y^2)} = \sqrt{\left(\frac{1 + \rho^2 - \rho^2 t^2}{\sigma^2}\right)(1 - 1 + t^2)} =$$

$$t \cdot \sqrt{\sigma^2 + \rho^2} \cdot \sqrt{1 - \frac{\rho^2 t^2}{\sigma^2 + \rho^2}}.$$

Let $1/\sqrt{\sigma^2 + \rho^2} = N$; let $\frac{\rho^2}{\sigma^2 + \rho^2} = k^2$;

also, $N = \frac{a - b}{4r\sqrt{ab}} \sqrt{4r^2 - (a - b)^2}$;

also, $K^2 = \frac{(a - b)^2\,[(a + b)^2 - 4r^2]}{16 r^2\, ab}$.

<u>The ρ^2 - Integral</u>. - Operating on $M \int_{-p}^{p} \rho^2 \frac{dy}{\sqrt{Y}}$ we have

$$2M \int_0^p \rho^2 \frac{dy}{\sqrt{Y}} = -2M \int_1^0 \frac{\rho^2}{\frac{t}{\sigma}\sqrt{\sigma^2 + \rho^2}\,\sqrt{1 - \frac{\rho^2 t^2}{\sigma^2 + \rho^2}}} \cdot \frac{t\, dt}{\sqrt{(1 - t^2)}}$$

$$= 2MN \int_0^1 \frac{\rho^2\, dt}{\sqrt{(1 - t^2)(1 - K^2 t^2)}}. \qquad (a)$$

Let $\sqrt{T} = \sqrt{(1 - t^2)(1 - k^2 t^2)}$ and write Eqn (a)

$$2MN\rho^2 \int_0^1 \frac{dt}{\sqrt{T}}, \qquad (b)$$

which integral we recognise as F (k, t), a standard Legendre form multiplied by the constant term 2 MN ρ^2. k^2 is known as the modulus and must be less than one. Obviously $k^2 = \frac{\rho^2}{\sigma^2+\rho^2} < 1$.

The A - Integral. - Consider now

$$M \int_{-p}^{p} \frac{A \, dy}{(1 - y^2)^2 \sqrt{Y}} = + 2 M \int_{0}^{p} \frac{A \, dy}{(1 - y^2)^2 \sqrt{Y}} . \qquad (a)$$

Putting $y^2 = \frac{1 - t^2}{\sigma^2}$ in $(1 - y^2)^2$ we get

$$(1 - y^2)^2 = \left(1 - \frac{1 - t^2}{\sigma^2}\right)^2 = \left(\frac{\sigma^2 - 1}{\sigma^4}\right)^2 \left[1 + \frac{1}{\sigma^2 - 1} t^2\right]^2.$$

Let $n \equiv \frac{\sigma^2 - 1}{\sigma^2}$. (also $n = \frac{(a - b)^2}{4 ab}$) .

Substituting these values in (a), we have (b)

$$2 M \int_{0}^{p} \frac{A \, dy}{(1 - y^2)^2 \sqrt{Y}} = 2 MNA \int_{0}^{1} \frac{dt}{(1 + nt^2)^2 \sqrt{T}} = \frac{4MN(\rho^2+1)\sigma^4 Z_2}{\sigma^2 - 1}.$$

Consider $\int_{0}^{1} \frac{dt}{(1 + nt^2)^2 \sqrt{T}}$. Z_ν defines $\int_{0}^{1} \frac{dt}{(1 + nt^2)^\nu \sqrt{T}}$, also

$$Z_1 = \int_{0}^{1} \frac{dt}{(1 + nt^2)\sqrt{T}} = \pi(n, k, t,), \text{ a standard form.}$$

Z_ν is reducible to Z_1 by the formula,

$$Z_\nu = \left(\frac{\left[\frac{t_1 \sqrt{T}}{1 + nt^2}\right]_0^t - r_1 Z_{\nu-1} - r_2 Z_{\nu-2} - r_3 Z_{\nu-3}}{r_0} \right). \qquad (c)$$

where

$$r_0 = \frac{(2\nu - 2)(n + 1)(n + k^2)}{n^2},$$

$$r_1 = - \frac{(2\nu - 3)[n(n + 2) + (2 n + 3) k^2]}{n^2},$$

$$r_2 = \frac{(2\nu - 4) [n + (n + 3) k^2]}{n^2},$$

$$r_3 = - \frac{(2\nu - 5) k^2}{n^2},$$

which may be used as many times as necessary. 2 is the power to which our $(1 + nt^2)$ binomial is raised, hence, $\nu = 2$. Let us evaluate r_0, r_1 , r_2 , r_3 . Using the value $\frac{(a - b)^2}{4ab}$ for n they are:

$$r_0 = \left\{\frac{(a+b)^4}{2\,r^2\,(a-b)^2}\right\},$$

$$r_1 = -\left\{\frac{(a+b)^2[(a+b)^2 + 2\,ab - 2\,r^2]}{2\,r^2\,(a-b)^2}\right\},$$

$$r_2 = 0,$$

$$r_3 = \frac{ab\,[(a+b)^2 - 4r^2]}{r^2\,(a-b)^2};$$

whence,

$$\frac{r_1}{r_0} = -\frac{[(a+b)^2 + 2\,ab - 2\,r^2]}{(a+b)^2},$$

and

$$\frac{r_3}{r_0} = \frac{2\,ab\,[(a+b)^2 - 4r^2]}{(a+b)^4}.$$

Therefore, by (c)

$$\frac{4\,MN(\rho^2+1)}{(\sigma^2-1)^2}\sigma^4\,Z_2 = \frac{4\,MN(\rho^2+1)\sigma^4}{(\sigma^2-1)^2\,r_0}\left(\left[\frac{t_1\sqrt{T}}{1+nt^2}\right]_0^1 - r_1 Z_1 - r_3 Z_3\right)$$

$$= \frac{4\,MN\,(\rho^2+1)\sigma^4}{(\sigma^2-1)^2}\left(-\frac{r_1}{r_0}\,\pi(n,k,t) - \frac{r_3}{r_0}\int_0^1 (1+nt^2)\,\frac{dt}{\sqrt{T}}\right)$$

$$= \frac{4\,MN\,(\rho^2+1)\sigma^4}{(\sigma^2-1)^2}\left(-\frac{r_1}{r_0}\,\pi(n,k,t) - \frac{r_3}{r_0}\left[F\,(k,t) + nY_\mu\right]\right) \qquad \text{(d)}$$

where $Y_\mu = \int_0^1 (t^2)^\mu \frac{dt}{\sqrt{T}}$. It is apparent $\mu = 1$ in (d).

$E\,(k,t) = \int_0^t \frac{dt}{\sqrt{T}} - k^2 \int_0^t t^2 \frac{dt}{\sqrt{T}}$ in general; whence,

$n\int_0^1 t^2 \frac{dt}{\sqrt{T}} = \frac{n}{k^2}[\,F(k,t) - E\,(k,t)]$. We substitute the value just obtained in (d) and get as our complete A - Integral:

$$\frac{4\,MN\,(\rho^2+1)\sigma^4}{(\sigma^2-1)^2}\left[\frac{r_1}{r_0}\left(\frac{n}{k^2}E(k,t) - \frac{n+k^2}{k^2}F\,(k,t)\right) - \frac{r_3}{r_0}\pi(n,k,t)\right]$$

The B - Integral. - It is easily seen:

$M\int_{-p}^{p}\frac{B}{(1-y)^2}\frac{dy}{\sqrt{Y}} = \frac{2\,MN\,(-1-3\rho^2)\sigma^2}{\sigma^2-1}\pi(n,k,t)$, since the result is immediately obtained by making the t - transformation and substituting the value of B.

The General Formula. - We obtain the general formula by adding the ρ^2 - , A -, and B - Integrals together, having first stated each integral in terms of the products of the parameters a, b, m, r, and the standard Legendre Elliptic Integrals. We find

$$S = 2r\sqrt{ab}\ \left[E(k,1) - F(k\ 1) + \frac{(a+b)^2}{4\ ab}\ \pi(n,k,1)\right]. \quad (\eta)$$

CHAPTER III

SPECIAL CASES

We come now to a consideration of our special cases.

Case i. - $l \neq 0$, $a = b$.

Our cone has become a right circular cone as fig. 2 shows. The general formula reduces to πra, which checks with the corresponding formula from elementary geometry.

$$S = 2r\sqrt{ab}\left[\int_0^1 \frac{\sqrt{1-k^2t^2}}{1-t^2}\,dt - \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} + \frac{(a+b)^2}{4ab}\int_0^1 \frac{dt}{(1+nt^2)\sqrt{(1-t^2)(1-k^2t^2)}}\right].$$

For this case, $k^2 = o$, $n = o$, which values cause the form to reduce to $S = 2ra\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \text{arc sin } \frac{t}{1}\Big|_0^1 \cdot 2ra = \pi ra$.

Case ii. - $l = o$, $a = b = r$.

Geometrically, our cone has degenerated to a circle. The general formula checks with the geometry, for the substitution of the values $k = o$, $n = o$, $a = b = r$ reduce it to $S = 2r^2\int_0 \frac{dt}{\sqrt{1-t^2}}$. Solving, we have

$S = 2r^2 \text{arc sin } t\Big|_0^1 = \pi r^2$. See fig. 3.

Case iii. - $l = o$, $a = r + m$, $b = r - m$.

If we observe fig. 4, we note that our cone has again degenerated to a circle. The vertex does not coincide with the center as before, but the elements do not cross. Considering our general formula, we find as before the substitution of $k = o$ cause our two first integrals to vanish. The general formula becomes.

$$S = 2r\sqrt{r^2 - m^2}\left[\frac{r^2}{r^2 - m^2}\int_0^1 \frac{dt}{(1+nt^2)(1-t^2)^{\frac{1}{2}}}\right].$$

Also, $S = 2r\sqrt{r^2 - m^2}\ (1 + n) \int_0^1 \frac{dt}{(1 + nt^2)(1 - t^2)}$.

Put $t = \sin\varphi$ and integrate.

$$S = 2r\sqrt{r^2 - m^2}\ (1 + n) \int_0^{\frac{\pi}{2}} \frac{d\varphi}{1 + n\sin^2\varphi} .$$

I. e., $S = -2r\sqrt{r^2 - m^2}\ (1 + n) \int_0^{\frac{\pi}{2}} \frac{-\csc^2\varphi\ d\varphi}{(1 + n) + \cot^2\varphi}$,

and $S = -\ 2r\sqrt{r^2 - m^2}. \left(\frac{r^2}{r^2 - m^2}\right) . \frac{\sqrt{r^2 - m^2}}{r} .\ \text{arc tan}\left[\frac{\cot\varphi}{\sqrt{1 + n}}\right]_0^{\frac{\pi}{2}}$.

Therefore, $S = \pi r^2$, which result verifies the geometrical interpretation.

<u>Case iv</u> . - $l = o$, $a = r + m$, $b = M - r$, $k^2 = 1$.

Interpretating this case geometrically, we must take the shaded area of fig. 5 twice. This is necessary in order to properly regard those elements of the cone proceeding from the arc B b C to the vertex V which are folded under the visible elements proceeding from arc BAC to V . Our general formula for this case becomes:

$$S = 2r\sqrt{m^2 - r^2}\ \left[\int_0^1 dt - \int_0^1 \frac{dt}{1 - t^2} + (1 + n) \int_0^1 \frac{dt}{(1+nt^2)(1-t^2)}\right] .$$

Integrating the last integral by partial fractions gives:

$$S = 2r\sqrt{m^2 - r^2}\ \left[1 - \int_0^1 \frac{dt}{1-t^2} + n \int_0^1 \frac{dt}{1+nt^2} + \int_0^1 \frac{dt}{1-t^2}\right]$$

$$= 2r\sqrt{m^2 - r^2}\ \left[1 + \int_0^1 \frac{dt}{\frac{1}{n} + t^2}\right] = 2r\sqrt{m^2 - r^2} + 2r^2 \tan^{-1} \frac{r}{\sqrt{m^2 - r^2}} . \quad \text{(a)}$$

S is found geometrically as follows:

$S = \pi r^2 + 2$ area ABCO $- 2$ area sector A b C O.

$$S = \pi r^2 + 2r\sqrt{m^2 - r^2} - 2r^2\ \text{arc tan} \frac{\sqrt{m^2 - r^2}}{r} . \quad \text{(b)}$$

Comparing (b) and (a), it is obvious $2r\sqrt{m^2 - r} = 2r\sqrt{m^2 - r}$.

Let us show that $\pi r^2 - 2r^2 \tan^{-1}\sqrt{m^2 - r^2} = 2r^2 \tan^{-1} \frac{r}{\sqrt{m^2 - r^2}}$.

Consulting the figure, it is apparent that the area of two equal sectors, Δ, Δ' , represented by $2\ r^2$ arc tan $\frac{\sqrt{m^2 - r^2}}{r}$ leaves two other equal sectors δ , δ'. To show that $\delta + \delta' = 2\delta' =$ $2\ r^2 \tan^{-1} \frac{r}{\sqrt{m^2 - r^2}}$.

$\frac{2 \text{ area } \delta}{\text{area } \theta\ (\pi r^2)} = \frac{2\ (\pi - 2\theta\ )}{2\pi}$, $\qquad 2 \text{ area } \delta = 2\ r^2 \left(\frac{\pi}{2} - \theta\right)$,

which is represented by $2\ r^2 \tan^{-1} \frac{r}{\sqrt{m^2 - r^2}}$.

Hence, the gen. formula is applicable to this case.

Case v . - $m = r$, $l = o$, $a = 2r$, $b = o$.

The substitution of these values in S would give an indeterminate value for the area of our cone. We evaluate S, then, by using case ii. Take $m > r$, $l = o$, and proceed to the limit as $m \to r$. We know that as $m \to r$, $a \to 2r$ and $b \to o$. Case ii gives S the value $2r \sqrt{m^2 - r^2} + 2r^2 \tan^{-1} \frac{r}{\sqrt{ab}}$.

Thus, as $m \to r$, $b \to o$ and $S \to 2r^2 \tan^{-1} \infty = \pi\ r^2$ in the limit. The circle of fig. 6 is the correct interpretation geometrically of this case, as the elements do not cross.

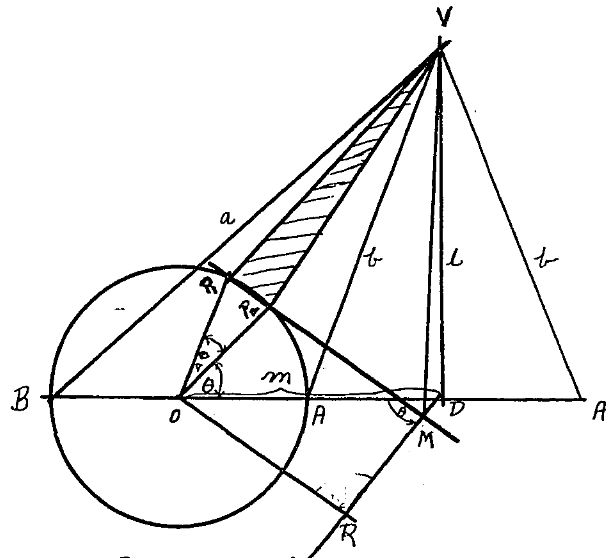

Fig. 1

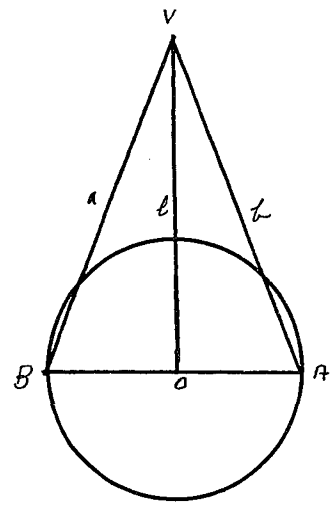

Fig. 2

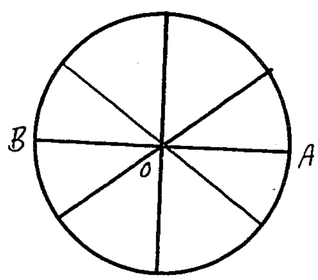

Fig. 3

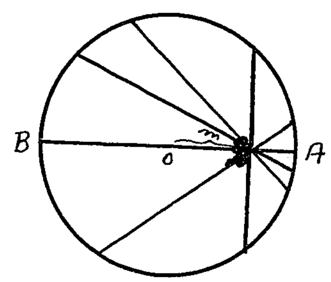

Fig. 4

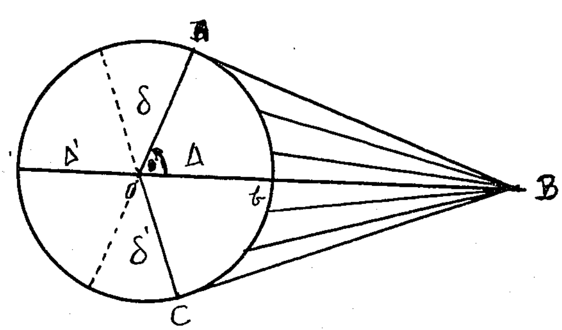

Fig. 5.

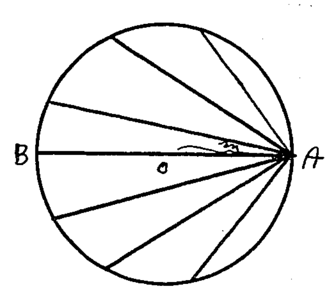

Fig. 6