ABSTRACT

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MATHEMATICAL ANALSIS OF A TRULY NONLINEAR

OSCILLATOR DIFFERENTIAL EQUATION

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We investigate the mathematical properties of solutions to the differential equation

$$\ddot{x} + x\ 1/3 = 0,$$

where $\ddot{x}$ indicates the second-derivative with respect to t. First, we demonstrate that all the solutions are periodic and then calculate an exact formula for the period. These two properties are derived from the behavior of the trajectories for this system in the (x,y) phase-space, where $y = \dot{x} = \frac{dx}{dt}$. We then use the methods of harmonic balance and iteration to determine approximations to both the periodic solutions and the corresponding periods. Our measure of the accuracy of these solutions is to use the concept of percentage-error. On this basis, we find that in general the harmonic balance methods produce better analytical approximations than those given by the iteration methods. Generalizations of our results to related research problems are given.

MATHEMATICAL ANALYSIS OF A TRULY NONLINEAR OSCILLATOR DIFFERENTIAL EQUATION

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First derivative: $\dot{x} \equiv dx/dt$

Second derivative: $\ddot{x} \equiv d^2x/dt^2$

HOH: higher-order harmonics

IC: initial condition

ODE: ordinary differential equation

TNL: truly nonlinear

CHAPTER 1
INTRODUCTION

The linear harmonic oscillator equation provides a mathematical approximation for the modeling of a broad range of phenomena in the natural and engineering sciences [1, 2, 3, 4]. This second-order differential equation is

$$\ddot{x} + x = 0, \tag{1.1}$$

where the following notation is used:

$$\begin{aligned} t &= \text{independent variable,} \\ x(t) &= \text{dependent variable,} \\ \dot{x} &= \text{first-derivative with respect to } t, \\ \ddot{x} &= \text{second-derivative with respect to } t. \end{aligned}$$

For the initial conditions (IC)

$$x(0) = A, \quad \dot{x}(0) = 0, \tag{1.2}$$

the solution to Eq. (1.1) is

$$x(t) = A\cos t, \tag{1.3}$$

and it follows that all solutions are periodic with period 2π.

The linear harmonic oscillator equation can be generalized to the form

$$\ddot{x} + f(x) = 0, \tag{1.4}$$

where the function $f(x)$ has the properties:

i) $f(0) = 0$,

ii) $f(-x) = -f(x)$,

iii) $xf(x)$ is monotonic increasing.

Under these conditions, all of the solutions to Eq. (1.4) are periodic [4].

If Eq. (1.4) can be rewritten in the form

$$\ddot{x} + x + \epsilon F(x) = 0, \tag{1.5}$$

where ϵ is a positive parameter, then it is called a nonlinear perturbed oscillator differential equation. For an arbitrary $F(x)$, Eq. (1.5) cannot be solved in terms of a finite number of elementary functions [5]. However, for small values of ϵ, i.e.,

$$0 < \epsilon \ll 1, \tag{1.6}$$

approximations to both the period and the periodic solutions may be calculated using techniques from the general theory of perturbations [4, 6, 7, 8, 9]. Since none of these methods are applicable to the differential equation we have investigated, no further discussion on this topic is required.

1.1 Research Problem

This dissertation provides a summary of our investigations on the differential equation

$$\ddot{x} + x^{1/3} = 0. \tag{1.1.1}$$

Inspection of this equation shows that it is a second-order, nonlinear ordinary differential equation (ODE). Further, this equation has no limiting form that corresponds to the harmonic oscillator differential equation

$$\ddot{x} + x = 0.$$

This fact implies that none of the standard methods can be applied to calculate approximations to the periodic solutions of Eq. (1.1.1). We will name Eq. (1.1.1) the cube-root oscillator.

The primary objective of our research was to answer the following questions and/or resolve certain related issues:

i) What are the general properties of the solutions to the cube-root differential equation?
ii) Can we prove that all solutions are periodic?
iii) Can the exact period of the periodic solutions (if they exist) be calculated?
iv) What calculational methods can be applied to the cube-root equation to determine analytical approximations to its periodic solutions?
v) Can a quantitative measure be formulated to assess the accuracy of the approximate solutions?

The general analysis and methodology used in this work is based on the prior research of Mickens [10, 11, 12, 13, 14]. It should also be noted that part of my work has appeared in an abstract [15] and one peer-reviewed publication [16].

1.2 Brief History of Fractional-Power Nonlinearities

The 2001 publication of Mickens [12] initiated the systematic study of nonlinear oscillators with fractional-power nonlinearities. These second-order differential equations take the form

$$\ddot{x} + x^{\frac{1}{2m+1}} = 0, \tag{1.2.1}$$

where m is a positive integer, i.e., $m = 1, 2, 3 \ldots$. A generalization of this equation is

$$\ddot{x} + x^{\frac{2n+1}{2m+1}} = 0, \tag{1.2.2}$$

where $n \neq m$ and (n, m) are now required to be non-negative integers. Note that for the cube-root equation

$$\ddot{x} + x^{1/3} = 0,$$

we have $n = 0$ and $m = 1$.

A further generalization was made by Gottlieb [17] in his study of the differential equation

$$\ddot{x} + |x|^p \text{sgn}(x) = 0, \tag{1.2.3}$$

with

$$0 < p < 1, \tag{1.2.4}$$

where $\text{sgn}(x)$ is defined to be

$$\text{sgn}(x) = \begin{cases} +1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

However, a detailed examination of Eq. (1.2.3) shows that this type of equation is valid for all $p > 0$. However, Gottlieb's work is of general interest since he did an analysis of all cases for which p takes rational values.

1.3 Outline of Presentation

In Chapter 2, we define and discuss concepts and calculational techniques that were used to analyze the cube-root differential equation. In particular, we introduce the idea of a "truly nonlinear (TNL)" oscillator equation and demonstrate that the cube-root equation belongs to this class of second-order, nonlinear differential equations. Other topics included in this chapter are the definition of a periodic function, odd-parity systems, and the elements of a phase-space analysis. We also present the basic methodology of approximation theory and use it to formulate the methods of harmonic balance and iterations. Finally, for completeness, we state several important trigonometric relations needed for the calculations and give a brief overview on the technique for solving second-order linear, inhomogeneous differential equations.

Chapter 3 contains details of our mathematical work. We first prove that all solutions of the cube-root differential equation are periodic and then use standard techniques to determine the exact value of the period. Next, we use two powerful methods to calculate approximations to the periodic solutions. Six different calculations are done.

The last chapter contains a summary of all results, presents a comparative analysis of the six solutions, and gives a brief discussion on how my particular research topics may be generalized.

CHAPTER 2
BACKGROUND INFORMATION

This chapter provides brief introductions and summaries of needed mathematical concepts and techniques required to carry out and analyze the calculations presented in Chapter 3. Further details regarding these topics are given in the indicated references.

2.1 Truly Nonlinear Oscillators

Consider a function $g(x)$, having the property

$$g(-x) = -g(x). \tag{2.1.1}$$

This function is defined to be a "truly nonlinear" (TNL) function at $x = 0$ if either of the following two conditions hold:

i) $g(0) = 0$ and $dg(0)/dt = 0$.

ii) Either $g(0)$ or $dg(0)/dt$ do not exist.

Particular examples of TNL functions are

$$g_1(x) = x^3, \quad g_2(x) = x^{1/3}.$$

For $g_1(x)$, we have $g_1(0) = 0$ and $dg_1(0)/dx = 0$, while for $g_2(x)$, we have

$$g_2(0) = 0, \quad \frac{dg_2(x)}{dx}\bigg|_{x=0} = \frac{1}{3}\left(\frac{1}{x^{2/3}}\right)\bigg|_{x=0} = \text{undefined}.$$

Therefore, both $g_1(x)$ and $g_2(x)$ are TNL functions.

A third, more interesting example is

$$g_3(x) = x + x^{1/3}.$$

Note that since

$$g_3(0) = 0, \quad \frac{dg_3(0)}{dx} = \left[1 + \left(\frac{1}{3}\right)\frac{1}{x^{2/3}}\right]\Bigg|_{x=0} = \text{undefined},$$

$g_3(x)$ is also a TNL function.

The second-order, nonlinear differential equation

$$\ddot{x} + g(x) = 0$$

is a TNL equation if $g(x)$ is a TNL function. Therefore,

$$\ddot{x} + x^3 = 0,$$
$$\ddot{x} + x + x^{1/3} = 0,$$
$$\ddot{x} + \frac{1}{x} = 0,$$

are all TNL differential equations. Further, the equation studied in our work,

$$\ddot{x} + x^{1/3} = 0, \tag{2.1.2}$$

is a TNL differential equation.

2.2 Odd-Parity Differential Equations

The general second-order, differential equation can be written as

$$F(x, \dot{x}, \ddot{x}) = 0. \tag{2.2.1}$$

Under the replacement $x \longrightarrow -x$, we have

$$\dot{x} \longrightarrow -\dot{x}, \quad \ddot{x} \longrightarrow -\ddot{x}.$$

If

$$F(-x, -\dot{x}, -\ddot{x}) = -F(x, \dot{x}, \ddot{x}), \tag{2.2.2}$$

then the differential equation is said to be an odd-parity equation.

The cube-root differential equation

$$\ddot{x} + x^{1/3} = 0,$$

is an odd-parity equation since

$$F(x, \dot{x}, \ddot{x}) \equiv \ddot{x} + x^{1/3} = 0,$$

and for $x \longrightarrow -x$, we find

$$\ddot{x} \longrightarrow -\ddot{x}, \quad x^{1/3} \longrightarrow (-x)^{1/3} = -x^{1/3}.$$

The result in the last equation is a consequence of the fact that $(-1)^{1/3}$ has a real root equal to (-1) [18].

2.3 Periodic Functions

Let $f(t)$ be defined on the interval, $-\infty < t < \infty$. Assume that a positive, fixed constant T exists such that for any t

$$f(t+T) = f(t). \tag{2.3.1}$$

This value of T is called a period of the function $f(t)$.

Let n be an integer, then it follows from this definition that

$$f(t+nT) = f(t).$$

The smallest value of T for which Eq. (2.3.1) holds is called the fundamental period for $f(t)$.

In general, periodic functions have the following properties [19, 20]:

i) Let $f(t)$ be a periodic function, with period T, then $cf(t)$, where c is an arbitrary constant, is also a periodic function with period T.

ii) Let $f_1(t)$ and $f_2(t)$ be periodic functions of period T, then $c_1 f_1(t) + c_2 f_2(t)$, where c_1 and c_2 are arbitrary constants, is a periodic function with period T.

iii) Let $f(x)$ be an integrable function with period T, then for any real constant c,

$$\int_c^{c+T} f(t)dt = \int_0^T f(t)dt.$$

For many applications, the period is not the relevant quantity needed to analyze the periodic behavior of the function $f(t)$. The new quantity usually introduced is the angular frequency. It is denoted by Ω and related to the period by the formula

$$\Omega = \frac{2\pi}{T}. \tag{2.3.2}$$

Therefore, a knowledge of either one allows the calculation of the other.

2.4 Fourier Series

The research presented in this dissertation is concerned with the periodic solutions of the TNL differential equation

$$\ddot{x} + x^{1/3} = 0, \tag{2.4.1}$$

subject to the initial conditions (IC)

$$x(0) = A, \quad \dot{x}(0) = 0. \tag{2.4.2}$$

It follows from the general theory of ordinary differential equations (ODE) that the solutions $x(t)$ have the following properties [21, 22]:

i) The solution, $x(t)$, and its first derivative, $\dot{x}(t)$, are continuous on the closed interval, $0 \leq t \leq T$, where T is the period.

ii) The period is a function of the IC, $x(0) = A$.

iii) With the IC's, given by Eq. (2.4.2), the solution $x(t)$ is an even function of t, i.e.,

$$x(-t) = x(t). \tag{2.4.3}$$

iv) The periodic solution, $x(t)$, has a Fourier series representation given by the expression

$$x(t) = \sum_{k=0}^{\infty} a_k \cos[(2k+1)\Omega t], \tag{2.4.4}$$

where the angular frequency $\Omega = 2\pi/T$.

Note that not all harmonics appear in Eq. (2.4.4), i.e., even harmonics, $2k\Omega$, are not present. This result is a consequence of the fact that Eq. (2.4.1) is an odd-parity differential equation and it has been shown by Mickens [23] that for this situation only odd harmonics can appear in the Fourier series representation.

2.5 Important Trigonometric Relations

The calculations of the next chapter make extensive use of trigonometric relations involving products of sine and cosine functions. Listed below are those formulas essential to our work [20]:

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2$$

$$\sin^2\theta = \left(\frac{1}{2}\right)(1-\cos 2\theta)$$
$$\cos^2\theta = \left(\frac{1}{2}\right)(1+\cos 2\theta)$$

$$(\sin\theta)^3 = \left(\frac{1}{4}\right)(3\sin\theta - \sin 3\theta)$$
$$(\cos\theta)^3 = \left(\frac{1}{4}\right)(3\cos\theta + \cos 3\theta)$$

$$\sin\theta_1\cos\theta_2 = \left(\frac{1}{2}\right)[\sin(\theta_1+\theta_2) + \sin(\theta_1-\theta_2)]$$
$$\cos\theta_1\sin\theta_2 = \left(\frac{1}{2}\right)[\sin(\theta_1+\theta_2) - \sin(\theta_1-\theta_2)]$$
$$\cos\theta_1\cos\theta_2 = \left(\frac{1}{2}\right)[\cos(\theta_1+\theta_2) + \cos(\theta_1-\theta_2)]$$
$$\sin\theta_1\sin\theta_2 = \left(\frac{1}{2}\right)[\cos(\theta_1-\theta_2) - \cos(\theta_1+\theta_2)]$$

$$\frac{d}{d\theta}\cos\theta = -\sin\theta$$
$$\frac{d}{d\theta}\sin\theta = \cos\theta$$
$$\int\cos\theta\,d\theta = \sin\theta$$
$$\int\sin\theta\,d\theta = -\cos\theta$$

2.6 Second-Order ODE's: Constant Coefficients

One of the approximation methods we use gives rise to second-order ODE's having constant coefficients and a periodic inhomogeneous term. These equations take the form

$$\ddot{y} + \Omega^2 y = \sum_{k=0}^{N} A_k \cos(2k+1)\Omega t, \tag{2.6.1}$$

where $(\Omega^2, A_1, A_2, \ldots, A_n)$ are given constants. The general solution can be written [20, 21, 22]

$$y(t) = y^{(H)}(t) + y^{(P)}(t), \tag{2.6.2}$$

where $y^{(H)}(t)$ is the homogeneous solution

$$y^{(H)}(t) = C_1 \cos(\Omega t) + C_2 \sin(\Omega t), \tag{2.6.3}$$

and the C_1 and C_2 are arbitrary constants, and the particular solution takes the form [20, 22]

$$y^{(P)}(t) = D_{11} t \cos(\Omega t) + D_{12} t \sin(\Omega t) + \sum_{k=1}^{N} D_k \cos(2k+1)\Omega t. \tag{2.6.4}$$

The constants $(D_{11}, D_{12}, D_1, D_2, \ldots, D_n)$ may be determined by substitution of $y^{(P)}(t)$ into the left-side of Eq. (2.6.1) and setting to zero the coefficients of the various linearly independent trigonometric terms. However, before carrying out this procedure, several restrictions and requirements should be noted:

i) For the IC's, $y(0) = A$ and $\dot{y}(0) = 0$, $y(t)$ must be an even function of t, see Section 2.4. This implies that, *a priori*, C_2 in Eq. (2.6.3), and D_{11}, in Eq. (2.6.4), must each be zero.

ii) Under the result in i), the homogeneous and particular solutions now become

$$y^{(H)}(t) = C_1 \cos(\Omega t), \tag{2.6.5}$$

$$y^{(P)}(t) = D_{12} t \sin(\Omega t) + \sum_{k=1}^{N} D_k \cos(2k+1)\Omega t. \tag{2.6.6}$$

However, the first term in $y^{(P)}(t)$ is not periodic; it is an oscillatory function having an amplitude that increases with t. Such an expression is called a *secular term*. Now D_{12} is given by the expression [20, 22]

$$D_{12} = \frac{A_1}{2\Omega}, \tag{2.6.7}$$

and it follows that the only way to have a periodic solution for $y(t)$ is to have $A_1 = 0$.

In the calculations to follow in Chapter 3, the coefficient A_1 will depend on the IC, $y(0) = A$, and an unknown angular frequency Ω, i.e.,

$$A_1 = A_1(A, \Omega). \tag{2.6.8}$$

The requirement of no secular terms gives

$$A_1(A, \Omega) = 0, \tag{2.6.9}$$

and the solution of this equation will give Ω as a function of A.

If the particular solution

$$y^{(P)}(t) = \sum_{k=1}^{N} D_k \cos(2k+1)\Omega t \tag{2.6.10}$$

is substituted into

$$\ddot{y} + \Omega^2 y = \sum_{k=1}^{N} A_k \cos(2k+1)\Omega t,$$

we obtain

$$-\Omega^2 \sum_{k=1}^{N} D_k[(2k+1)^2 - 1] \cos(2k+1)\Omega t = \sum_{k=1}^{N} A_k \cos(2k+1)\Omega t$$

and

$$D_k = \frac{A_k}{\Omega^2[1-(2k+1)^2]}, \quad k = 1, 2, \ldots, n. \tag{2.6.11}$$

Therefore, $y(t)$ is

$$y(t) = y^{(H)}(t) + y^{(P)}(t) = C_1 \cos \Omega t + \sum_{k=1}^{N} \frac{A_k \cos(2k+1)\Omega t}{\Omega^2[1-(2k+1)^2]}. \tag{2.6.12}$$

Applying the initial condition, $y(0) = A$, gives

$$A = C_1 + \sum_{k=1}^{N} \frac{A_k}{\Omega^2[1-(2k+1)^2]},$$

and if this is solved for C_1, we find

$$C_1 = A - \sum_{k=1}^{N} \frac{A_k}{\Omega^2[1-(2k+1)^2]}. \tag{2.6.13}$$

Thus, in this manner, the complete, periodic solution to the linear, second-order, inhomogeneous ODE given by Eq. (2.6.1) is constructed.

2.7 Phase-Space

The second-order ODE

$$\frac{d^2x}{dt^2} + g(x) = 0, \tag{2.7.1}$$

can be rewritten as two first-order coupled ODE's

$$\frac{dx}{dy} = y, \quad \frac{dy}{dt} = -g(x). \tag{2.7.2}$$

In this form it is useful to introduce two-dimensional space, (x, y), and study the properties of the solutions to Eq. (2.7.1) in this 2-dim geometry. (See Mickens [20] and Liu [22] for full discussions of this topic.)

The trajectories in the (x, y) "phase-space" are denoted by $y = y(x)$ and are determined by solutions to a first-order ODE. This equation can be derived by noting that if $y = y(x)$, then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \tag{2.7.3}$$

and

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{g(x)}{y}. \tag{2.7.4}$$

The result in Eq. (2.7.3) is a consequence of the implicit differentiation rule; see Section 2.3, in [24]. Equation (2.7.4) is called the *trajectory equation* for the two first-order ODE's given by Eq. (2.7.2).

Constant solutions to Eqs. (2.7.2) are called fixed-points and are the (real) simultaneous solutions to the equations

$$\bar{y} = 0, \quad g(\bar{x}) = 0. \tag{2.7.5}$$

Note that for Eq. (2.7.1), all fixed-points have $\bar{y} = 0$.

Inspection of Eq. (2.7.4) shows that it is separable, i.e., it can be written as

$$y\,dy + g(x)dx = 0.$$

If this equation is integrated using the IC's

$$x(0) = A, \quad y(0) = 0, \tag{2.7.6}$$

then the following expression, called the *first-integral*, is found

$$\frac{y^2}{2} + V(x) = V(A), \tag{2.7.7}$$

where

$$V(x) = \int^x g(z)dz. \tag{2.7.8}$$

In physics and engineering, $V(x)$ is called the potential function [25], and $H(x, y)$, defined as

$$H(x, y) = \frac{y^2}{2} + V(x) = \text{constant}, \tag{2.7.9}$$

is called the energy function of the system modeled by the differential equation given in Eq. (2.7.1).

A direct consequence of Eq. (2.7.9) is that for every allowable constant C, the solutions to

$$H(x, y) = C,$$

correspond to a trajectory curve in the (x, y) phase-space [20, 21, 25].

The first-integral, $H(x, y)$, may be invariant under a transformation of the phase-space coordinates x and y. If such a transformation exists, then the system is said to have a symmetry. Examples of elementary symmetries include the following three:

(i) $x \longrightarrow -x$, $y \longrightarrow y$

This corresponds to reflection in the y-axis.

(ii) $x \longrightarrow x$, $y \longrightarrow -y$

This is reflection through the x-axis.

(iii) $x \longrightarrow -x$, $y \longrightarrow -y$

This corresponds to inversion or reflection through the origin.

The above three symmetries are usually denoted, respectively, by the symbols S_1, S_2, and S_3 [20].

If $g(-x) = -g(x)$, i.e., Eq. (2.7.1) is of odd-parity, then

$$\frac{dy}{dx} = -\frac{g(x)}{x}$$

is invariant under all three transformations.

There are two curves in the (x, y) phase-space that have special significance; they are the x- and y-nullclines, and are denoted, respectively, by $y_\infty(x)$ and $y_0(x)$. They are defined in the following manner:

i) The x-nullcline is a curve in the (x, y) phase-space along which $dy/dx = \infty$, i.e., whenever a solution trajectory crosses this curve, the slope of the solution is unbounded.

ii) The y-nullcline is a curve in the (x, y) phase-space along which $dy/dx = 0$, i.e., whenever a solution trajectory crosses the y-nullcline, the slope of the solution is zero.

Note, in general the x- and y-nullclines do not correspond to solution trajectories. Also, these nullclines divide the phase-space into several open domains where the boundaries of these domains are the nullclines themselves. In each open domain the "sign" of dy/dx is fixed, i.e., within a given domain dy/dx is bounded and is everywhere either negative or positive. The derivative, dy/dx, can only change sign by crossing from one open domain through a nullcline into another open domain.

Examination of Eq. (2.7.4) allows us to reach the following conclusions:

i) The x-nullclines consist of the vertical lines corresponding to the real solutions of $g(\bar{x}) = 0$.

ii) The y-nullcline is the x-axis.

2.8 Basic Approximation Methodology

The next two sections discuss two methods which can be applied to TNL oscillator equations to construct approximations for the periodic solutions. This section gives a general overview of the basic methodology and philosophy of constructing approximations to nonlinear equations based on the work of Mickens [26].

Let x be a function of some variable (which does not, at this point, have to be explicitly indicated). Assume that x is obtained from solving a nonlinear equation

$$N(x) = 0. \tag{2.8.1}$$

Further assume that to determine x from Eq. (2.8.1) certain restrictions must be placed on x, i.e.,

$$R : \{R_1(x), \ldots R_M(x)\}, \quad M \geq 1. \tag{2.8.2}$$

An example of this type of problem is

$$\ddot{x} + \frac{x^3}{1+x^2} = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{2.8.3}$$

where

$$N(x) \equiv \ddot{x} + \frac{x^3}{1+x^2},$$

and the two restrictions, in this case IC's, are

$$R_1(x) : x(0) = A, \tag{2.8.4}$$

$$R_2(x) : \dot{x}(0) = 0. \tag{2.8.5}$$

General strategies for constructing approximations to the solutions of the problem given by Eqs. (2.8.1) and (2.8.2) are to proceed in one of the following two ways:

(I) The nonlinear equation, $N(x) = 0$, is replaced by a set of equations that can be solved exactly.

For this case, the procedure replaces a single nonlinear equation by an infinite set of linear, inhomogeneous exactly solvable equations such that they are collectively solved by means of an iteration process. Symbolically, this corresponds to

$$\begin{gathered}\begin{bmatrix} N(x) = 0 \\ R : \{R_1(x), R_2(x), \dots, R_M(x)\} \end{bmatrix} \\ \Downarrow \\ \begin{bmatrix} Lx_{k+1} = F_k(x_0, \dots, x_k) \\ k = 0, 1, 2, \dots, \\ x_0 = \text{specified}, \\ R : \{R_1(x_{k+1}, R_2(x_{k+1}), \dots, R_M(x_{k+1})\} \end{bmatrix}, \end{gathered} \tag{2.8.6}$$

where $Lx = 0$ is a linear equation. Note that each x_k must satisfy the M-restrictions imposed on the solutions to $N(x) = 0$. However, it may be further required to impose additional restrictions on the solutions x_k.

To illustrate this last point, consider the TNL oscillator equation given by Eq. (2.8.3). It can be demonstrated that all its solutions are periodic for any set of IC's. However, the solutions to the iteration process may have both periodic and nonperiodic behaviors, and therefore some constraint must be placed on the calculated solutions such that they are restricted to only the periodic solutions.

The input or generating solution, x_0, is the basis upon which the iteration method is founded. Generally, it is selected to be the simplest function satisfying the M-restrictions of the original nonlinear problem, $N(x) = 0$.

(II) A second strategy for obtaining approximate solutions to Eqs. (2.8.1) and (2.8.2) is to assume that there exists a complete, orthogonal set of functions in which the solution x can be expanded. (For ODE's these functions correspond to solutions of Sturm-Liouville problems. For full details see Chapter 6 of Mickens [20] or Chapter 12 of Ross [22].)

Denote this infinite set of functions by $\{\phi_k\}$, $k = \{0, 1, 2, \dots\}$. The particular functions selected will be determined by the M-restrictions. Now assume that the exact solution to

$N(x) = 0$ has the representation

$$x = \sum_{k=0}^{\infty} c_k \phi_k, \tag{2.8.7}$$

where the $\{c_k\}$, $k = \{0, 1, 2, \ldots\}$ are constants. The L-th approximation to x is the expression

$$x_L = \sum_{k=0}^{L} \bar{c}_k \phi_k, \tag{2.8.8}$$

where the $\bar{c}_k$ are to be determined using some appropriate mathematical procedure. One way to do this is to substitute Eq. (2.8.8) into Eq. (2.8.1) and carry out the required mathematical operations to obtain the result

$$N(x_L) = \sum_{k=0}^{L} H_k(\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_L)\phi_k + \text{(higher-order-terms)} \simeq 0. \tag{2.8.9}$$

Since the $\{\phi_k\}$ are linearly independent [20, 21], it follows that

$$H_k(\bar{c}_0, \bar{c}_1, \ldots, c_L) = 0, \quad k = (0, 1, 2, \ldots, L), \tag{2.8.10}$$

and these $(L+1)$ equations may be solved for the $(L+1)$ coefficients in Eq. (2.8.8).

Note that both strategies assume that all the required mathematical operations can be carried out and that the calculated solutions are the ones appropriate for the original nonlinear problem, $N(x) = 0$.

2.9 Harmonic Balance

The method of harmonic balance is a powerful method for calculating approximations to the periodic solutions of nonlinear oscillator differential equations. A detailed discussion of this procedure, its applications, and its limitations is given in Chapter 4 of the book by Mickens [4]. We present below a summary of the method of harmonic balance.

Assume that the following ODE

$$\ddot{x} + g(x) = 0, \tag{2.9.1}$$

is of odd-parity, i.e.,

$$g(-x) = -g(x), \tag{2.9.2}$$

and the IC's are selected to be

$$x(0) = A, \quad \dot{x}(0) = 0. \tag{2.9.3}$$

Further assume that all the solutions to Eq. (2.9.1) are periodic. The latter condition, along with the requirement of odd-parity, implies that $x(t)$ has a Fourier representation of the form

$$x(t) = \sum_{m=1}^{\infty} A_m \cos(2m-1)\Omega t, \tag{2.9.4}$$

where the coefficients and angular frequency are expected to depend on A, i.e.,

$$A_m = A_m(A), \quad m = (1, 2, 3, \dots); \quad \Omega = \Omega(A).$$

The N-th order method of harmonic balance approximates $x(t)$ by the function

$$x_N(t) = \sum_{m=1}^{N} \bar{A}_m^{(N)} \cos[(2m-1)\bar{\Omega}_N t], \tag{2.9.5}$$

where the coefficients $\{\bar{A}_m^{(N)}\}$, $m = (1, 2, \dots, N)$, are approximations to $\{A_m\}$, $m = (1, 2, \dots, N)$; and $\bar{\Omega}_N$ is an approximation to Ω.

Examination of Eq. (2.9.5) shows that there are $(N+1)$ unknowns: the N coefficients and $\bar{\Omega}_N$. These quantities are calculated in the following manner:

i) Substitute Eq. (2.9.5) into Eq. (2.9.1) and write the resulting expression as a linear combination of cosine functions; but, only include harmonics to order $(2N-1)$. Carrying out this calculation gives

$$\sum_{m=1}^{N} H_m \cos[(2m-1)\bar{\Omega}_N t] + \text{HOH} \simeq 0 \tag{2.9.6}$$

where HOH means "higher-order harmonics," and the H_m are functions of all the coefficients and $\bar{\Omega}_N$, i.e.,

$$H_m \equiv H_m(\bar{A}_1^{(N)}, \bar{A}_2^{(N)}, \ldots, \bar{A}_N^N, \bar{\Omega}_N), \quad m = (1, 2, \ldots, N). \tag{2.9.7}$$

Note that for each function $g(x)$, the H_m are defined as unique functions of the coefficients and the N-th approximation to the angular frequency, $\bar{\Omega}_N$.

ii) Since the cosine functions are linearly independent [4], the results in Eq. (2.9.6) imply that the coefficients must be zero, i.e.,

$$H_m(\bar{A}_1^{(N)}, \bar{A}_2^{(N)}, \ldots, \bar{A}_N^N, \bar{\Omega}_N) = 0, \quad m = (1, 2, 3, \ldots, N). \tag{2.9.8}$$

iii) Solve the N functional relations, given by Eq. (2.9.8) for $(\bar{A}_2^{(N)}, \bar{A}_2^{(N)}, \ldots, \bar{A}_N^N, \bar{\Omega}_N)$ in terms of the coefficient $\bar{A}_1^N$, i.e.,

$$\bar{\Omega}_N = \bar{\Omega}_N(\bar{A}_1^{(N)}), \quad \bar{A}_m^{(N)} = \bar{A}_m^{(N)}(\bar{A}_1^{(N)}), \quad m = (2, 3, \ldots, N). \tag{2.9.9}$$

iv) Express $\bar{A}_1^{(N)}$ in terms of the initial condition by using the requirement

$$x_N(0) = A = \sum_{m=1}^{N} \bar{A}_m^{(N)} = \bar{A}_1^{(N)} + \sum_{m=1}^{N} \bar{A}_m^{(N)},$$

and from this it follows that

$$\bar{A}_1^{(N)} = A - \sum_{m=2}^{N} \bar{A}_m^{(N)}(\bar{A}_1^{(N)}). \tag{2.9.10}$$

This equation can be solved for $\bar{A}_1^{(N)}$ in terms of A.

Note that the IC, $\dot{x}_N(0) = 0$, is automatically satisfied by $x_N(t)$; see Eq. (2.9.5).

Finally, the substitution of Eq. (2.9.10) into Eq. (2.9.9), and the placement of these relations into Eq. (2.9.5), gives the N-th harmonic balance approximation to the periodic solution of Eq. (2.9.1)

2.10 Iteration

The iteration method for TNL oscillator differential equations was created by Mickens [11, 13] to analyze this class of equations for the case where the nonlinear term contains the dependent variable raised to a fractional power. However, the general technique can be applied to both standard nonlinear and TNL oscillator equations and many papers have been written on its applications to such differential equations [14, 20, 30, 31].

As in the previous section, we assume that the following ODE

$$\ddot{x} + g(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{2.10.1}$$

is of odd-parity and all solutions are periodic. Let us rewrite Eq. (2.10.1) to the form

$$\ddot{x} + \Omega^2 x = \Omega^2 x - g(x), \tag{2.10.2}$$

where Ω is a real parameter. The iteration method assumes that suitable approximations to the periodic solutions for Eq. (2.10.1) may be determined by solving the following recursive relations

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \Omega_k^2 x_k - g(x_k), \tag{2.10.3}$$

where the generating solution is

$$x_0(t) = A\cos(\Omega_0 t), \tag{2.10.4}$$

and for each value of k, we have the IC's

$$x_k(0) = A, \quad \dot{x}_k(0) = 0. \tag{2.10.5}$$

The angular frequency Ω_k^2 is unknown, but may be determined by a calculation at the $(k+1)$ level. Since the homogeneous equation

$$\ddot{x}_{k+1}^{(H)} + \Omega_k^2 x_{k+1}^{(H)} = 0, \tag{2.10.6}$$

has the solution

$$x_{k+1}^{(H)}(t) = c_1 \cos(\Omega_k t), \quad c_1 = \text{arbitrary constant}, \tag{2.10.7}$$

Ω_k is determined by the requirement that terms, on the right-side of Eq. (2.10.3), which can give rise to secular terms, be eliminated from the particular solution, $x_{k+1}^{(P)}(t)$.

The general solution can be written as

$$x_{k+1}(t) = x_{k+1}^{(H)}(t) + x_{k+1}^{(P)}, \tag{2.10.8}$$

and the constant c_1 calculated from the IC, $x_{k+1}(0) = A$.

For the next level of calculation, $k + 2$, the Ω_k is replaced by Ω_{k+1} and the above procedure is repeated. At each level the Ω_k has a definite functional dependency on A, but this dependency changes as k varies.

CHAPTER 3
CALCULATIONS

This chapter presents all of my work on the cube-root oscillator differential equation

$$\ddot{x} + x^{1/3} = 0, \quad x(0) = A, \quad \dot{x}(0) = 0.$$

We first provide two independent proofs that all solutions to this differential equation are periodic. Next, we calculate an exact value for the angular frequency using the properties of an integral related to the beta function. The next two sections give the details of the calculations leading to several expressions for approximations of the periodic solutions for the cube-root oscillator. The procedures used to determine these approximate solutions are based on the methods of harmonic balance and iteration.

3.1 Proofs of Periodicity

The cube-root equation

$$\ddot{x} + x^{1/3} = 0; \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{3.1.9}$$

has the following representation in terms of two first-order, coupled ODE's

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x^{1/3}; \quad x(0) = A, \quad y(0) = 0. \tag{3.1.10}$$

The fixed-point or constant solution is $(\bar{x}, \bar{y}) = (0, 0)$, and the first-order ODE for the trajectories, $y = y(x)$, in the (x, y) phase-space is

$$\frac{dy}{dx} = -\frac{x^{1/3}}{y}. \tag{3.1.11}$$

This equation is separable and can be integrated to produce the following result for the first-integral

$$\frac{y^2}{2} + \left(\frac{3}{4}\right) x^{4/3} = \left(\frac{3}{4}\right) A^{4/3}. \tag{3.1.12}$$

As Eq. (3.1.12) represents a simple, closed curve for any value of $|A| > 0$ [4, 27], and since closed curves in the (x, y) phase-space correspond to periodic solutions, the result given by this equation implies that all the solutions of the cube-root TNL oscillator differential equations are periodic.

We can also produce a second proof of the periodicity of the solutions. To do this, note that the x- and y-nullclines, for Eq. (3.1.11), are

$$\frac{dy}{dx} = \infty : \text{the } x\text{-axis}; \tag{3.1.13}$$

$$\frac{dy}{dx} = 0 : \text{the } y\text{-axis}. \tag{3.1.14}$$

Inspection of Eq. (3.1.11) shows that the ODE is invariant under the following three transformation:

$$S_1 : x \to -x, \quad y \to y,$$

$$S_2 : x \to x, \quad y \to -y,$$

$$S_3 : x \to -x, \quad y \to -y,$$

where S_1 is reflection in the y-axis, S_2 is reflection in the x-axis, and S_3 is inversion through the origin.

The geometric proof that the cube-root differential equation has all periodic solutions proceeds as follows; see Figure 3.1:

i) Part (a) of Figure 3.1 presents the two null-clines, indicated by the vertical-dashed and horizontal-dashed lines. The x-nullcline corresponds to the x-axis and the y-

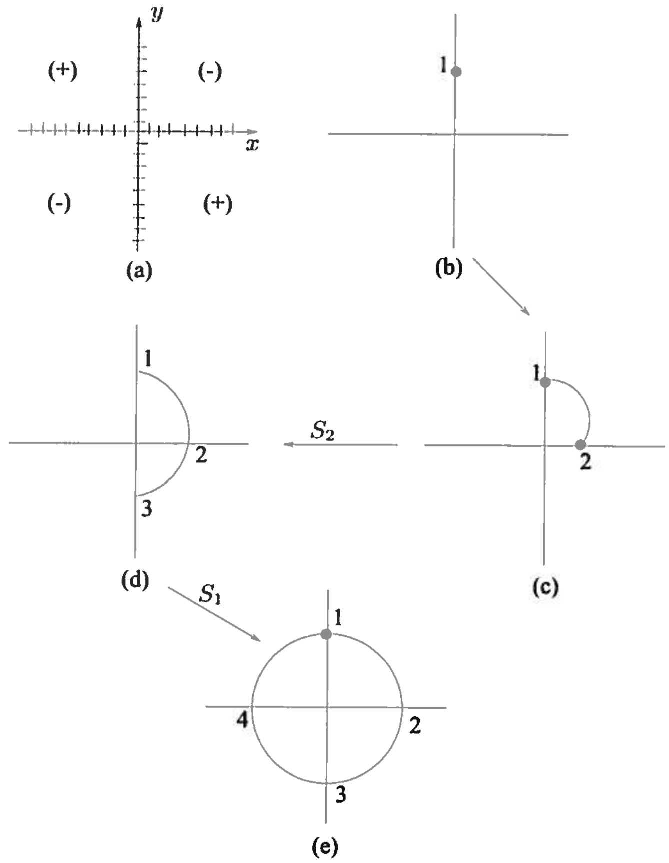

Figure 3.1: A geometric proof of periodicity for the cube-root ODE.

nullcline is the y-axis. The $(\pm)$ notation indicates the sign of dy/dx in each of the four domains into which the x- and y-nullclines divide the (x, y) phase plane.

ii) Select an arbitrary point in the (x, y) plane. For convenience, we take this point, P_1, to lie on the positive y-axis.

iii) Through P_1 draw the portion of the trajectory that lies in the first quadrant. Note that at P_1, the slope of this curve is zero. The slope of the curve in the first quadrant is negative until it contacts the x-axis and at this point, P_2, the slope is "minus infinity" or vertical to the x-axis. These actions produce the curve shown in Figure 3.1(c).

iv) Now apply the symmetry transformation S_2 and obtain the curve -1 2 3 indicated in

Figure 3.1(d).

v) The application of S_1 now gives the closed curve $-1\ 2\ 3\ 4\ 1$. However a closed curve implies the corresponding trajectory is a periodic solution. Therefore since the point, P_1, is arbitrary, it follows that all solutions must be periodic [26, 27].

The above geometrical analysis has been used extensively by Mickens in his work on nonlinear oscillator differential equations [4, 10, 11, 12, 13, 14, 15, 23, 28].

In summary, two different methods have been used to prove that all solutions to the cube-root TNL oscillator ODE are periodic. We now calculate the exact value of the period for this oscillator.

3.2 Exact Period of the Cube-Root Oscillator

From Eq. (3.1.12), the first-integral for the cube-root oscillator, we have

$$y^2 = \left(\frac{3}{2}\right)\left[A^{4/3} - x^{4/3}\right], \tag{3.2.1}$$

and from this it follows that

$$y = \frac{dx}{dt} = \pm\sqrt{\frac{3}{2}}\left[A^{4/3} - x^{4/3}\right]^{1/2}. \tag{3.2.2}$$

Now in the (x, y) plane, the motion along a trajectory from $x = A$ to $x = 0$ corresponds to that portion of the trajectory in the fourth quadrant and for this situation $y = dx/dt < 0$; see Figure 3.1(e). Therefore

$$\frac{dx}{dt} = -\sqrt{\frac{3}{2}}\left[A^{4/3} - x^{4/3}\right]^{1/2}$$

and

$$dt = -\left(\sqrt{\frac{2}{3}}\right)\frac{dx}{\left[A^{4/3} - x^{4/3}\right]^{1/2}}. \tag{3.2.3}$$

The existence of the three symmetry transformations implies that the time for the oscillator to go from $x = A$ to $x = 0$ is $T/4$, where T is the period of the oscillation. Therefore, integrating the expression in Eq. (3.2.3) gives

$$\int_0^{T/4} dt = -\left(\sqrt{\frac{2}{3}}\right) \int_A^0 \frac{dx}{[A^{4/3} - x^{4/3}]^{1/2}}$$

and

$$\frac{T}{4} = \left(\sqrt{\frac{2}{3}}\right) \int_0^A \frac{dx}{[A^{4/3} - x^{4/3}]^{1/2}} \,. \tag{3.2.4}$$

To obtain this last expression, we used the fact that

$$-\int_A^0 (\cdots)dx = \int_0^A (\cdots)dx.$$

Let us now make the transformation of variable

$$x = Au \tag{3.2.5}$$

in the integral given by Eq. (3.2.4); carrying out this change gives

$$dx = A\,du,$$

$$x = 0 \Rightarrow u = 0,$$

$$x = A \Rightarrow u = 1,$$

$$[A^{4/3} - x^{4/3}]^{1/2} = A^{2/3}[1 - u^{4/3}]^{1/2},$$

and

$$\int_0^A \frac{dx}{[A^{4/3} - x^{4/3}]^{1/2}} = A^{1/3} \int_0^1 \frac{du}{[1 - u^{4/3}]^{1/2}} \,. \tag{3.2.6}$$

Now let

$$w = u^{4/3} \Rightarrow u = w^{3/4}.$$

Therefore,

$$du = \left(\frac{3}{4}\right)\frac{dw}{w^{1/4}},$$
$$u = 0 \Rightarrow w = 0,$$
$$u = 1 \Rightarrow w = 1,$$

and

$$\int_0^1 \frac{du}{[1-u^{4/3}]^{1/2}} = \left(\frac{3}{4}\right)\int_0^1 w^{-1/4}(1-w)^{-1/2}dw. \tag{3.2.7}$$

Combining the results in Eqs. (3.2.4), (3.2.5) and (3.2.7) gives the expression

$$T(A) = \sqrt{6}A^{1/3}\int_0^1 w^{-1/4}(1-w)^{-1/2}dw. \tag{3.2.8}$$

The beta function, $B(p,q)$, is defined as [20]

$$B(p,q) \equiv \int_0^1 w^{q-1}(1-w)^{p-1}dw.$$

It can also be written in terms of the gamma function, $\Gamma(z)$, by means of the formula [20]

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Comparison of the integral, in Eq. (3.2.8), with the integral definition of the beta function, gives

$$q - 1 = -\frac{1}{4} \quad \text{or} \quad q = \frac{3}{4},$$
$$p - 1 = -\frac{1}{2} \quad \text{or} \quad p = \frac{1}{2}.$$

Therefore, the period of the cube-root TNL oscillator is

$$T(A) = \sqrt{6}A^{1/3}B\left(\frac{1}{2},\frac{3}{4}\right) = \left(\sqrt{6}A^{1/3}\right)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}. \tag{3.2.9}$$

From the Abramowitz and Stegun [29], we find the following values for the gamma functions

$$\Gamma\left(\frac{1}{2}\right) = 1.77245\,38509\ldots,$$
$$\Gamma\left(\frac{3}{4}\right) = 1.22541\,67024\ldots,$$
$$\Gamma\left(\frac{5}{4}\right) = \left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)(3.62560\,99082\ldots),$$

and therefore $T(A)$ is

$$T(A) = (A^{1/3})\left(4\sqrt{6}\right)\left[\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right] = (5.869664426)A^{1/3}. \tag{3.2.10}$$

The corresponding angular frequency, $\Omega(A)$, is

$$\Omega(A) = \frac{2\pi}{T(A)} = \frac{1.070450515}{A^{1/3}}. \tag{3.2.11}$$

Note that we have written the period, $T(A)$, and the angular frequency, $\Omega(A)$ to indicate that they both depend on the value of x at $t = 0$, i.e., $x(0) = A$.

3.3 Harmonic Balance

The method of harmonic balance can be applied to the cube-root oscillator differential equation

$$\ddot{x} + x^{1/3} = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{3.3.1}$$

The first-order, direct approximation includes only a single harmonic and takes the form

$$x_{DHB}(t) = A\cos(\Omega_{DHB}t) = A\cos\theta, \tag{3.3.2}$$

where

$$\theta \equiv \Omega_{DHB}t. \tag{3.3.3}$$

Note that $x_{DHB}(t)$ satisfies the IC's and Ω_{DHB} must be calculated. (We use "DHB" to indicate that this is a direct harmonic balance procedure.) Substitution of $x_{DHB}(t)$ into Eq. (3.3.1) gives

$$-(\Omega_{DHB}^2 A\cos\theta) + (A\cos\theta)^{1/3} \simeq 0. \qquad (3.3.4)$$

It has been shown by Mickens and Wilkins [20] that

$$(\cos\theta)^{1/3} = a_1\left[\cos\theta - \frac{\cos(3\theta)}{5} + \frac{\cos(5\theta)}{10} + \cdots\right], \qquad (3.3.5)$$

where

$$a_1 = 1.1595952669639\ldots. \qquad (3.3.6)$$

Using this result, Eq. (3.3.4) becomes

$$\left[-\Omega_{DHB}^2 A + a_1 A^{1/3}\right]\cos\theta + \text{HOH} \simeq 0, \qquad (3.3.7)$$

and setting the coefficient of $\cos\theta$ to zero gives

$$-\Omega_{DHB}^2 A + a_1 A^{1/3} = 0$$

or

$$\Omega_{DHB} = \frac{\sqrt{a_1}}{A^{1/3}} = \frac{1.076845}{A^{1/3}}. \qquad (3.3.8)$$

As a consequence of Eqs. (3.3.2) and (3.3.8), the first-order, direct harmonic balance approximation to the periodic solution of the TNL cube-root oscillator equation is

$$x_{DHB}(t) = A\cos\left[\left(\frac{\sqrt{a_1}}{A^{1/3}}\right)t\right]. \qquad (3.3.9)$$

A second first-order harmonic balance approximation to the periodic solution can be calculated by eliminating the cube-root term in Eq. (3.3.1), i.e.,

$$
\begin{aligned}
\ddot{x} + x^{1/3} &= 0 \\
\ddot{x} &= -x^{1/3} \\
(\ddot{x})^3 &= -x \\
(\ddot{x})^3 + x &= 0.
\end{aligned}
\tag{3.3.10}
$$

If we denote the first-order approximation by

$$
\begin{cases} x_{HB}^{(1)}(t) = A\cos\theta, \\ \theta \equiv \Omega_{HB}^{(1)} t, \end{cases}
\tag{3.3.11}
$$

then substitution of these expressions into Eq. (3.3.10) gives

$$
\left[-\left(\Omega_{HB}^{(1)}\right)^2 A\cos\theta \right] + A\cos\theta \simeq 0.
\tag{3.3.12}
$$

Using

$$
(\cos\theta)^3 = \left(\frac{3}{4}\right)\cos\theta + \left(\frac{1}{4}\right)\cos 3\theta,
$$

Eq. (3.3.12) becomes

$$
\left[-\left(\frac{3}{4}\right)(\Omega_{HB}^{(1)})^6 A^3 + A \right] \cos\theta + \mathrm{HOH} \simeq 0.
\tag{3.3.13}
$$

Setting the coefficient of $\cos\theta$ to zero and solving for the angular frequency gives

$$
\Omega_{HB}^{(1)} = \left(\frac{4}{3}\right)^{1/6} \left(\frac{1}{A^{1/3}}\right) = \frac{1.049115}{A^{1/3}}.
\tag{3.3.14}
$$

Therefore, the corresponding harmonic balance first-order solution is

$$
x_{HB}^{(1)}(t) = A\cos\left[\left(\frac{4}{3}\right)^{1/6} \frac{t}{A^{1/3}} \right].
\tag{3.3.15}
$$

The cube-root equation, in the form presented by Eq. (3.3.10), can be used to calculate a second-order harmonic balance approximation to its periodic solution. To do this, write this approximation to the solution as

$$\begin{cases} x_{HB}^{(2)}(t) = A_1 \cos\theta + A_2 \cos 3\theta \\ \theta = \Omega_{HB}^{(2)} t. \end{cases} \tag{3.3.16}$$

If a new variable, z, is introduced, i.e.,

$$A_2 = zA_1,$$

then $x_{HB}^{(2)}(t)$ becomes

$$x_{HB}^{(2)}(t) = A_1(\cos\theta + z\cos 3\theta). \tag{3.3.17}$$

The task is to determine A_1, z, and $\Omega_{HB}^{(2)}$ in terms of A. To do this, note that

$$\ddot{x}_{HB}^{(2)}(t) = -(\Omega_{HB}^{(2)})^2 A_1(\cos\theta + 9z\cos 3\theta). \tag{3.3.18}$$

Therefore, substituting Eqs. (3.3.17) and (3.3.18) into Eq. (3.3.10) gives

$$-(\Omega_{HB}^{(2)})^2 A_1^3(\cos\theta + 9z\cos 3\theta)^3 + A_1(\cos\theta + z\cos 3\theta) \simeq 0. \tag{3.3.19}$$

To proceed, we must evaluate $(\cos\theta + 9z\cos 3\theta)^3$, but only retain the $\cos\theta$ and $\cos 3\theta$ harmonics. We now give this calculation:

$$\begin{aligned} (\cos\theta + 9z\cos 3\theta)^3 &= (\cos\theta)^3 + 3(\cos\theta)^2(9z\cos 3\theta) \\ &\quad + 3(\cos\theta)(9z\cos 3\theta)^2 + (9z\cos 3\theta)^3 \\ &= \left[\left(\frac{3}{4}\right) + \left(\frac{27}{4}\right)z + \left(\frac{243}{2}\right)z^2\right]\cos\theta \\ &\quad + \left[\left(\frac{1}{4}\right) + \left(\frac{27}{2}\right)z + \left(\frac{2187}{4}\right)z^3\right]\cos 3\theta + \text{HOH}. \end{aligned} \tag{3.3.20}$$

Substituting this expression into Eq. (3.3.19) and collecting together the terms common to $\cos\theta$ and $\cos 3\theta$, we find

$$\begin{aligned} &\left\{-(\Omega_{HB}^{(2)})^6 A_1^3\left[\left(\frac{3}{4}\right) + \left(\frac{27}{4}\right)z + \left(\frac{243}{2}\right)z^2\right] - A_1\right\}\cos\theta \\ &\quad + \left\{-(\Omega_{HB}^{(2)})^6 A_1^3\left[\left(\frac{1}{4}\right) + \left(\frac{27}{2}\right)z + \left(\frac{2187}{4}\right)z^3\right] - zA_1\right\}\cos 3\theta \\ &\quad + \text{HOH} \simeq 0. \end{aligned} \tag{3.3.21}$$

Setting, respectively, the coefficients of $\cos\theta$ and $\cos 3\theta$ to zero gives the two expressions

$$(\Omega_{HB}^{(2)})^6 A_1^2 \left[\left(\frac{3}{4}\right) + \left(\frac{27}{4}\right) z + \left(\frac{243}{2}\right) z^2 \right] = 1, \tag{3.3.22}$$

$$(\Omega_{HB}^{(2)})^6 A_1^2 \left[\left(\frac{1}{4}\right) + \left(\frac{27}{2}\right) z + \left(\frac{2187}{4}\right) z^3 \right] = z. \tag{3.3.23}$$

A single equation for z can be derived by dividing Eqs. (3.3.22) by (3.3.23), and then simplifying the resulting expression. Carrying out this calculation gives

$$(1701)z^3 - (27)z^2 + (51)z + 1 = 0. \tag{3.3.24}$$

Since the harmonic balance procedure holds under the assumption that z must be small in magnitude [4], we only need to see if Eq. (3.3.24) has such a root. A rough calculation produces [4]

$$z \simeq -\left(\frac{1}{51}\right) = -0.019608,$$

while an accurate numerical solution of Eq. (3.3.24) gives

$$z = -0.019178. \tag{3.3.25}$$

Solving Eq. (3.3.22) for $\Omega_{HB}^{(2)}$ and using the IC

$$A_1(1+z) = A,$$

or

$$A_1 = \frac{A}{1+z}, \tag{3.3.26}$$

we find

$$\Omega_{HB}^{(2)} = \left[\frac{1+2z+z^2}{\left(\frac{3}{4}\right) + \left(\frac{27}{4}\right) z + \left(\frac{243}{2}\right) z^2} \right]^{1/6} \left(\frac{1}{A^{1/3}}\right) = \left(\frac{4}{3}\right)^{1/6} \left(\frac{1}{A^{1/3}}\right) g(z), \tag{3.3.27}$$

where

$$g(z) = \left[\frac{1+2z+z^2}{1+9z+162z^2} \right]^{1/6}. \tag{3.3.28}$$

Comparing Eqs. (3.3.14) and (3.3.27) gives

$$\Omega_{HB}^{(2)} = \Omega_{HB}^{(1)} g(z). \tag{3.3.29}$$

If $g(z)$ is evaluated with z from Eq. (3.3.25), then

$$\Omega_{HB}^{(2)} = \frac{1.063410}{A^{1/3}}. \tag{3.3.30}$$

Therefore, the second-order harmonic balance solution for the cube-root TNL oscillator equation is

$$x_{HB}^{(2)}(t) = \left(\frac{A}{1+z}\right)(\cos\theta + z\cos 3\theta), \tag{3.3.31}$$

with $\theta = \Omega_{HB}^{(2)} t$ and z given, respectively, by Eqs. (3.3.30) and (3.3.25).

3.4 Iteration

To apply an iteration procedure to the cube-root TNL equation, we must express it in the form

$$x = -(\ddot{x})^3. \tag{3.4.1}$$

Multiplying both sides by Ω^2 and then adding $\ddot{x}$ to both sides gives

$$\ddot{x} + \Omega^2 x = \ddot{x} - \Omega^2(\ddot{x})^3.$$

From this expression the following iteration scheme can be formulated

$$\begin{cases} \ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \ddot{x}_k - \Omega_k^2(\ddot{x}_k)^2, \\ x_0(t) = A\cos(\Omega_0 t), \\ x_k(0) = A, \quad \dot{x}_k(0) = 0; \quad k = 1, 2, \dots . \end{cases} \tag{3.4.2}$$

Let $\theta = \Omega_0 t$; then for $k = 0$, we have

$$
\begin{aligned}
\ddot{x}_1 + \Omega_0^2 x_1 &= \ddot{x}_0 - \Omega_0^2(\ddot{x}_0)^3 \\
&= (-\Omega_0^2 A\cos\theta) - \Omega_0^2(-\Omega_0^2 A\cos\theta)^3 \\
&= -\Omega_0^2 A\cos\theta + \Omega_0^8 A^3\left[\left(\frac{3}{4}\right)\cos\theta + \left(\frac{1}{4}\right)\cos 3\theta\right] \\
&= \left[-\Omega_0^2 A + \left(\frac{3}{4}\right)\Omega_2^8 A^3\right]\cos\theta + \left(\frac{\Omega_0^8 A^3}{4}\right)\cos 3\theta. \qquad (3.4.3)
\end{aligned}
$$

To obtain a periodic solution for $x_1(t)$, the solutions must not contain secular terms. However, the secular terms may be eliminated by setting the coefficient of the $\cos\theta$ term to zero on the right-side of Eq. (3.4.3). If this is done, we obtain

$$
-\Omega_0^2 A\left[1 - \left(\frac{3}{4}\right)\Omega_0^6 A^2\right] = 0,
$$

and solving for Ω_0 gives

$$
\Omega_0 = \left(\frac{4}{3}\right)^{1/6}\frac{1}{A^{1/3}} = \frac{1.049115}{A^{1/3}}. \qquad (3.4.4)
$$

Therefore,

$$
x_0(t) = A\cos\left[\left(\frac{4}{3}\right)^{1/6}\left(\frac{t}{A^{1/3}}\right)\right]. \qquad (3.4.5)
$$

The determination of $x_1(t)$ requires finding the solution to the following second-order, linear, inhomogeneous ODE

$$
\ddot{x}_1 + \Omega_0^2 x_1 = \left(\frac{\Omega_0^8 A^3}{4}\right)\cos 3\theta, \qquad (3.4.6)
$$

where $\theta = \Omega_0 t$. The homogeneous solution is [21]

$$
x_1^{(H)}(t) = C_1\cos\theta, \qquad (3.4.7)
$$

where C_1 is an arbitrary constant. The particular solution is [22]

$$
x_1^{(P)}(t) = D_1\cos 3\theta. \qquad (3.4.8)
$$

Using the fact that

$$\ddot{x}_1^{(P)}(t) = -9\Omega_0^2 D_1 \cos 3\theta, \tag{3.4.9}$$

and substituting Eqs. (3.4.8) and (3.4.9) into Eq. (3.4.6), we find the result

$$-9\Omega_1^2 D_1 \cos 3\theta + D_1 \cos 3\theta = \left(\frac{\Omega_0^8 A^3}{4}\right) \cos 3\theta$$

or

$$D_1 = -\left(\frac{1}{8\Omega_0^2}\right)\left(\frac{\Omega_0^8 A^3}{4}\right) = -\frac{\Omega_0^6 A^3}{32} = \left[\left(\frac{4}{3}\right)\frac{1}{A^2}\right]\left(\frac{A^3}{32}\right) = -\left(\frac{A}{24}\right), \tag{3.4.10}$$

and

$$x_1^{(P)}(t) = -\left(\frac{A}{24}\right) \cos 3\theta. \tag{3.4.11}$$

Since the general solution is

$$x_1(t) = x_1^{(H)}(t) + x_1^{(P)}(t) = C_1 \cos\theta - \left(\frac{A}{24}\right) \cos\theta, \tag{3.4.12}$$

and $x_1(t)$ must satisfy the IC

$$x_1(0) = A, \tag{3.4.13}$$

it follows that

$$A = C_1 - \frac{A}{24}$$

or

$$C_1 = \left(\frac{25}{24}\right) A,$$

and

$$x_1(t) = A\left[\left(\frac{25}{24}\right)\cos\theta - \left(\frac{1}{24}\right)\cos 3\theta\right]. \tag{3.4.14}$$

If we stop at $k = 0$, then $x_1(t)$ is given by Eq. (3.4.14) with

$$\theta = \left(\frac{4}{3}\right)^{1/6}\left(\frac{t}{A^{1/3}}\right). \tag{3.4.15}$$

For $k = 1$, we have

$$\ddot{x}_2 + \Omega_1^2 x_2 = \ddot{x}_1 - \Omega_1^2 (\ddot{x}_1)^3, \tag{3.4.16}$$

and

$$x_1(t) = A[\alpha \cos\theta - \beta \cos 3\theta], \tag{3.4.17}$$

$$\theta = \Omega_1 t, \quad \alpha = \frac{25}{24}, \quad \beta = \frac{1}{24}, \tag{3.4.18}$$

where Ω_1 must be determined by solving Eq. (3.4.16). Substituting Eq. (3.4.17) into the right-side of Eq. (3.4.16) and carrying out the required mathematical operations gives

$$\ddot{x}_2 + \Omega_1^2 x_2 = -\Omega_1^2 \left[\alpha - \left(\frac{3A^2}{4}\right) \Omega_1^6 h(\alpha, \beta)\right] A \cos\theta + \text{HOH}, \tag{3.4.19}$$

where

$$h(\alpha, \beta) = (\alpha^2 - \alpha\beta + 2\beta^2)\alpha. \tag{3.4.20}$$

(We have not written down the HOH terms since only Ω_1 is to be calculated, and not $x_1(t)$.) The absence of secular terms in the solution for $x_2(t)$ requires that the coefficients of the $\cos\theta$ term be zero, i.e.,

$$\Omega_1^2 \left[\alpha - \left(\frac{3A^2}{4}\right) \Omega_1^6 h(\alpha, \beta)\right] A = 0, \tag{3.4.21}$$

or

$$\Omega_1 = \left[\left(\frac{4}{3}\right)^{1/6} \frac{1}{A^{1/3}}\right] \left[\frac{1}{h(\alpha, \beta)}\right]^{1/6} = \Omega_0 \left[\frac{1}{h(\alpha, \beta)}\right]^{1/6} = \frac{1.041424}{A^{1/3}}. \tag{3.4.22}$$

Therefore, the evaluation of $x_1(t)$, by calculating Ω_1 from the next level iteration is

$$\begin{cases} x_1(t) = A\left[\left(\frac{25}{24}\right)\cos\theta - \left(\frac{1}{24}\right)\cos 3\theta\right], \\ \theta = \Omega_1 t. \end{cases} \tag{3.4.23}$$

An iteration scheme that allows (currently) only one step of iteration was constructed by Mickens [14]. It starts with the original cube-root equation

$$\ddot{x} + x^{1/3} = 0,$$

and formulates the iteration scheme as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \Omega_k^2 x_k - x_k^{1/3}. \tag{3.4.24}$$

The main difficulty with this scheme is that while

$$(x_0)^{1/3} = (A\cos\theta)^{1/3} = A^{1/3}(\cos\theta)^{1/3}$$

can be expanded into an infinite set of cosine functions [4], higher levels of iteration require a knowledge of the Fourier series for the function

$$f(\theta) = [c_1\cos\theta + c_2\cos 3\theta + \cdots]^{1/3}, \tag{3.4.25}$$

where c_1, c_2, etc., are constants, and currently it is not known how to achieve this goal [14]. In other words, it follows that the general theorems on Fourier series [4, 19] allow us to conclude that $f(\theta)$ has the representation

$$f(\theta) = d_1\cos\theta + d_2\cos 3\theta + \cdots, \tag{3.4.26}$$

but no general procedure is known such that the coefficients d_1, d_2, etc., can be calculated.

The following is a concise summary of Mickens' calculations [14].

For $k = 0$, the iteration equation for $x_1(t)$ is

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 x_0 - x_0^{1/3}. \tag{3.4.27}$$

Using $x_0(t) = A\cos\theta$, where $\theta = \Omega_0 t$, we find

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2(A\cos\theta) - (A\cos\theta)^{1/3}.$$

Now $(\cos\theta)^{1/3}$ has the Fourier representation [3]

$$(\cos\theta)^{1/3} = \sum_{n=0}^{\infty} a_{2n+1}\cos(2n+1)\theta,$$

where

$$a_{2n+1} = \frac{3\Gamma\left(\frac{7}{3}\right)}{2^{4/3}\Gamma\left(n+\frac{5}{3}\right)\Gamma\left(\frac{2}{3}-n\right)},$$

and a_1 has the value given in Eq. (3.3.6). Making this substitution for $(\cos\theta)^{1/3}$ into the right-side of Eq. (3.4.27), the following expression is obtained

$$\ddot{x}_1 + \Omega_0^2 x_1 = (\Omega_0^2 A - A^{1/3} a_1)\cos\theta - A^{1/3}\sum_{n=1}^{\infty} a_{2n+1}\cos(2n+1)\theta. \tag{3.4.28}$$

The absence of secular terms requires the coefficient of the $\cos\theta$ term to be zero, i.e.,

$$\Omega_0^2 A - A^{1/3} a_1 = 0 \tag{3.4.29}$$

or

$$\Omega_0 = \frac{\sqrt{a_1}}{A^{1/3}} = \frac{1.076845}{A^{1/3}}. \tag{3.4.30}$$

Therefore, the function $x_1(t)$ satisfies the ODE

$$\ddot{x}_1 + \Omega_0^2 x_1 = -A^{1/3}\sum_{n=1}^{\infty} a_{2n+1}\cos(2n+1)\theta$$

and the complete solution to it is [14]

$$x_1(t) = \beta A\cos(\Omega_0 t) + A\sum_{n=1}^{\infty}\left\{\frac{a_{2n+1}}{a_1[(2n+1)^2-1]}\right\}\cos[(2n+1)\Omega_0 t], \tag{3.4.31}$$

where the constant β is

$$\beta = 1 - \sum_{n=1}^{\infty}\frac{a_{2n+1}}{a_1[(2n+1)^2-1]}. \tag{3.4.32}$$

Inspection of Eq. (3.4.31) shows that all harmonics appear in Mickens' calculation of $x_1(t)$.

CHAPTER 4
ANALYSIS AND RESEARCH EXTENSIONS

In Chapter 3, we calculated five approximations to the periodic solutions of the cube-root TNL oscillator. Three solutions were obtained using the method of harmonic balance and two others were derived from application of the iteration technique. To judge the accuracy of these solutions, we use a measure based on the calculation of the percentage error for the angular frequency. This particular analysis of the accuracy of a solution has been used successfully by Mickens [13, 14] and other researchers [30, 31]

Let Ω_{exact} be the exact value of the angular frequency and let Ω be the value determined from a calculation. The percentage-error of the calculated value is defined to be

$$\% \text{ error in } \Omega \equiv \left| \frac{\Omega_{\text{exact}} - \Omega}{\Omega_{\text{exact}}} \right| \cdot 100\%.$$

4.1 Analysis

Tables 4.1 and 4.2, respectively, provide a summary of our calculations for the harmonic balance and iteration methods. The following is an analysis of the results contained in these two tables.

1) The calculated approximate values for the angular frequencies all have the mathematical structure

$$\Omega(A) = \frac{C}{A^{1/3}}. \tag{4.1.1}$$

The value of C is dependent on the particular method used to determine $\Omega(A)$. Since the

Table 4.1: Harmonic Balance Solutions

	$x(t)$*	$A^{1/3}\Omega(A)$**	% error in $\Omega(A)$
DHB	$A\cos(\Omega_{DHB}t)$ Eq. (3.3.9)	1.076845 Eq. (3.3.8)	0.60%
HB-1	$A\cos(\Omega_{HB}^{(1)}t)$ Eq. (3.3.15)	1.049115 Eq. (3.3.14)	1.99%
HB-2	$\left(\frac{A}{1+z}\right)[\cos\theta + z\cos 3\theta]$ $z = -0.019178$ $\theta = \Omega_{HB}^{(2)}t$ Eqs. (3.3.25) and (3.3.31)	1.063410 Eq. (3.3.30)	0.66%

*The angular frequency for a given solution is given in the third column.
** $\Omega_{\text{exact}}(A) = 1.070450515$

period is

$$T(A) = \frac{2\pi}{\Omega(A)} = \left(\frac{2\pi}{C}\right) A^{1/3}, \tag{4.1.2}$$

it follows that $T(A)$ increases with an increase of A. However, this increase is very slow since the amplitude A is raised to the one-third power.

2) The direct harmonic balance (DHB) calculation gives a very accurate solution, i.e., 0.60% error. Note that the magnitude of this error is essentially the same as that gotten from the second-order harmonic balance procedure applied to the rationalized form of

$$\ddot{x} + x^{1/3} = 0, \tag{4.1.3}$$

i.e.,

$$x = -(\ddot{x})^3. \tag{4.1.4}$$

This solution is denoted BH-2 in Table 4.1.

Table 4.2: Iteration Solutions

	$x(t)$*	$A^{1/3}\Omega(A)$**	% error in $\Omega(A)$
I_0	$A\cos(\Omega_0 t)$ Eq. (3.4.5)	1.049115 Eq. (3.4.4)	1.99%
I_1	$A\left[\left(\frac{25}{24}\right)\cos\theta - \left(\frac{1}{24}\right)\cos 3\theta\right]$ $\theta = \Omega_0 t$ Eq. (3.4.14)	1.049115 Eq. (3.4.4)	1.99%
I_1'	$A\left[\left(\frac{25}{24}\right)\cos\theta - \left(\frac{1}{24}\right)\cos 3\theta\right]$ $\theta = \Omega_1 t$ Eq. (3.4.14)	1.041424 Eq. (3.4.22)	2.71%

*The angular frequency for a given solution is given in the third column.
** $\Omega_{\text{exact}}(A) = 1.070450515$

3) The second-order harmonic balance solution, i.e., HB-2, is approximately three times more accurate than the first-order harmonic balance solution, HB-1. This result is consistent with the expectation that higher order harmonic balance calculations give more accurate solutions than those of lower order.

4) While our harmonic balance calculations were restricted to first- and second-orders, the inclusion of higher order harmonics is a natural consequence of using ever increasing orders of the harmonic balance method. However, the coefficients of the higher harmonics decrease rapidly with the order of the harmonics [4].

5) None of the calculations based on iteration methods give solutions as accurate as those determined from the use of harmonic balance methods. In fact, the iteration method derived solution appears to lose accuracy as the order of the calculation increases.

4.2 Conclusion

Based on the analysis, presented in Section 4.1, the following conclusion can be reached:

Harmonic balance related methods, in general, give more accurate results in comparison to iteration techniques.

However, the major difficulty with the harmonic balance method is that its use gives rise to coupled, cubic algebraic equations. These equations have, in general, no exact solutions expressible in simple closed forms.

Finally, it should be emphasized that the cube-root TNL equation, given by Eq. (4.1.3) must be re-expressed in the rationalized form, given by Eq. (4.1.4), to obtain higher order calculations for both the harmonic balance and iteration procedures.

4.3 Summary

We have investigated the mathematical properties of the cube-root TNL differential equation. The following properties were derived:

1) There is a unique fixed-point in the (x, y) phase-space located at $(\bar{x}, \bar{y}) = (0, 0)$.
2) The first-integral for this equation is

$$\frac{y^2}{2} + \left(\frac{3}{4}\right) x^{4/3} = \left(\frac{3}{4}\right) A^{4/3}, \tag{4.3.1}$$

   where $x(0) = A$ and $y(0) = \dot{x}(0) = 0$.
3) The mathematical structure of the first-integral implies that all solutions are periodic.
4) From the first-order system equations

$$\dot{x} = y, \quad \dot{y} = -x^{1/3}, \tag{4.3.2}$$

it follows that the trajectory curves, $y = y(x)$, in the (x, y) phase-space satisfy the differential equation

$$\frac{dy}{dx} = -\frac{x^{1/3}}{y}. \tag{4.3.3}$$

Note that the first-integral, Eq. (4.3.1) is the solution to Eq. (4.3.3).

5) From either Eq. (4.3.1) or Eq. (4.3.3), it follows that the trajectory curves, in the (x, y) phase-space, are invariant under the following three transformations:

$$S_1 : x \to -x, \quad y \to +y,$$

$$S_2 : x \to +x, \quad y \to -y,$$

$$S_3 : x \to -x, \quad y \to -y.$$

6) The symmetry transformations, given in 5), allow us to again conclude that all solutions to the cube-root TNL equation are periodic.

Finally, we used the above stated properties and the methods of harmonic balance and iteration to calculate first- and second-order approximations for the angular frequency and solution to the cube-root TNL equation.

4.4 Research Extensions

The research results given in this dissertation may be extended by considering the following important issues:

1) Is it possible to formulate the method of harmonic balance such that the calculation of the angular frequency and the amplitudes only involve the solution of linear algebraic equations?

2) Are there ways of constructing iteration procedures such that a higher order iteration calculation produces a more accurate solution than a lower order determination?

3) Can a procedure be formulated that combines the essential features of both the harmonic balance and iteration methods, yet gives at each level of the calculation only linear algebraic and differential equations to be solved?

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