On vector application to plane analytic geometry

Jean Marie Wright  
Atlanta University

Follow this and additional works at: http://digitalcommons.auctr.edu/dissertations
Part of the Physical Sciences and Mathematics Commons

Recommended Citation
ON VECTOR APPLICATION

TO

PLANE ANALYTIC GEOMETRY

A THESIS
SUBMITTED TO THE FACULTY OF ATLANTA UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF MASTER OF SCIENCE

BY

JEAN MARIE WRIGHT

DEPARTMENT OF MATHEMATICS

ATLANTA, GEORGIA

JUNE, 1958
ACKNOWLEDGMENTS

For tireless assistance and guiding inspiration the writer is deeply indebted to Dr. Lonnie Cross. Without his assistance, the completion of this work would not have been possible.

Appreciation also goes to Mrs. G. C. Smith who also served as a guide during the graduate career of the writer.
TABLE OF CONTENTS

ACKNOWLEDGMENTS ........................................... ii

LIST OF FIGURES ............................................. iii

Chapter

I. INTRODUCTION ........................................... 1

II. BASIC DEFINITIONS ....................................... 2
   Vector .................................................. 2
   The Zero Vector ........................................ 3
   Scalar ................................................... 3
   Magnitude ............................................... 3
   Equality .................................................. 3
   Scalar Times a Vector .................................. 3
   Collinear and Coplanar Vectors ....................... 4

III. FUNDAMENTAL OPERATIONS ON VECTORS ................. 5
   Addition of Vectors .................................... 5
   Subtraction of Vectors ................................ 7
   Components of Vectors ................................ 8
   The Unit Vectors ....................................... 9
   Scalar or Dot Product of Two Vectors ............... 10
   Vector Product of Two Vectors ....................... 11

IV. APPLICATION TO PLANE ANALYTIC GEOMETRY ............. 13
   Distance of a Point from a Line ..................... 13
   Division of a Line Segment in a Given Ratio ........ 15
   The Distance Between Two Points .................... 16
   The Area of a Triangle ................................ 17
   Vector Equation of a Circle ......................... 18
   Vector Equation of the Tangent to the Circle ....... 19
   Pencils of Circles ..................................... 19
   Power of a Point and Radical Axis ................... 24
   A Circle Orthogonal to a Family ..................... 26
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Geometric Representation of a Vector</td>
<td>2</td>
</tr>
<tr>
<td>2.</td>
<td>Vector from the Origin to Any Point on a Line</td>
<td>3</td>
</tr>
<tr>
<td>3.</td>
<td>Collinear Vectors</td>
<td>4</td>
</tr>
<tr>
<td>4.</td>
<td>Addition of Vectors</td>
<td>5</td>
</tr>
<tr>
<td>5.</td>
<td>Associativity of Vectors</td>
<td>6</td>
</tr>
<tr>
<td>6.</td>
<td>Similar Triangles</td>
<td>7</td>
</tr>
<tr>
<td>7.</td>
<td>Subtraction of Vectors</td>
<td>8</td>
</tr>
<tr>
<td>8.</td>
<td>Components of Vectors</td>
<td>8</td>
</tr>
<tr>
<td>9.</td>
<td>Distance from the Origin in Terms of Rectangular Coordinates</td>
<td>9</td>
</tr>
<tr>
<td>10.</td>
<td>Vector Product</td>
<td>11</td>
</tr>
<tr>
<td>11.</td>
<td>Distance of a Point from a Line</td>
<td>13</td>
</tr>
<tr>
<td>12.</td>
<td>The Perpendicular Distance from a Point to a Line</td>
<td>14</td>
</tr>
<tr>
<td>13.</td>
<td>The Division of a Line Segment in a Given Ratio</td>
<td>15</td>
</tr>
<tr>
<td>14.</td>
<td>The Distance Between Two Points</td>
<td>16</td>
</tr>
<tr>
<td>15.</td>
<td>The Area of a Triangle</td>
<td>17</td>
</tr>
<tr>
<td>16.</td>
<td>A circle with Radius</td>
<td>18</td>
</tr>
<tr>
<td>17.</td>
<td>The Tangent Line to a Circle</td>
<td>19</td>
</tr>
<tr>
<td>18.</td>
<td>Line Circles</td>
<td>22</td>
</tr>
<tr>
<td>19.</td>
<td>Line Circle of a Pencil</td>
<td>24</td>
</tr>
<tr>
<td>20.</td>
<td>Orthogonal Circles</td>
<td>25</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

Vector analysis is a recent development. It was preceded by quaternions originated by Sir William R. Hamilton in 1843 and by H. G. Grassmann in 1844. We owe a large part of the development of vectors to two mathematical physicists, Joseph Willard Gibbs, 1881–84, and Oliver Heaviside, 1891.

The desire was to develop a system of vector analysis, the operational rules of which would conform with the corresponding rules of scalar algebra.

Certain physical quantities, called scalar quantities are expressed by specifying their magnitudes. Examples of these are time, electric change and energy. Others, however, require the specification of direction as well. Examples of these are velocity, force and displacement of a point which are called vector quantities or simply vectors.

The problem of this study is to relate the concept of vectors to certain concepts in plane analytic geometry. Chapter II has to do with certain basic definitions in connection with vectors. Chapter III deals with the fundamental operations, an understanding of which is necessary for application. Chapter IV has the desideratum of relating some of the foregoing concepts to some concepts in plane analytic Geometry.
CHAPTER II

BASIC DEFINITIONS

1. Vector: A vector is an ordered pair of points; a quantity having both magnitude and direction. It may be represented by letters which designate the end points of a segment $AB$ or by one letter $v$ which is the line that determines the vector. An example of the geometric representation of a vector is

\[ \overrightarrow{AB}. \]

FIGURE I

where $A$ represents the beginning or initial point and $B$ represents the end or terminal point.

As a matter of convention, throughout this work, capital letters will represent points and small letters will represent lines. A pair of capital letters with a line or a small letter with a line will denote the vector.
2. **The Zero Vector:** The vector with magnitude zero represents the zero vector. It can be denoted by the degenerated line segment in which the initial point corresponds to the terminal point. It is parallel and perpendicular to every vector in the plane.

3. **Scalar:** A scalar is a quantity possessing magnitude only. The value of a scalar remains constant under all systems of reference. Familiar scalar quantities are the real numbers.

4. **Magnitude:** The numerical length of a vector $\mathbf{v}$ is called its magnitude. This number is represented by $|\mathbf{v}|$. If $|\mathbf{v}| = 0$, $\mathbf{v}$ is the zero or null vector.

5. **Equality:** Any two vectors are equal if they have the same magnitude and direction. The family of all parallel vectors in the plane having the same magnitude and direction are equal.

6. **Scalar times a Vector:** Multiplying a scalar times a vector affects the magnitude, the direction or both magnitude and direction of a vector. Thus $c\mathbf{v}$ is a vector whose direction is the same as $\mathbf{v}$ but whose magnitude is $c$-times $\mathbf{v}$. $-c\mathbf{v}$ is a vector whose magnitude is $c$-times $\mathbf{v}$ but whose direction is the reverse of that of $\mathbf{v}$.

![Figure 2](image-url)
Consider the vector $\vec{a}$ in Figure 2. The vector $\vec{v}$ drawn from the origin to any point of the line $OA$ in either direction is $\vec{v} = xa$.

This represents, with $x$ as a variable scalar, the vector equation of all points in the line $OA$.

7. **Collinear and Coplanar Vectors:** Vectors which are parallel to the same line are collinear. The family of all vectors parallel to a fixed vector in a plane represents collinear vectors (figure 3).

The family of all vectors which belong to the same plane are called coplanar.
8. **Addition of Vectors:** Any two vectors may generate a third by defining the addition of two in the following manner:

![Diagram showing vector addition]

**FIGURE 4**

From figure 4, \( \overrightarrow{AB} = \vec{a}, \overrightarrow{BC} = \vec{b} \). The vector \( \vec{c} = \overrightarrow{AC} \) is the result of \( \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \). By construction \( \overrightarrow{AB} = \overrightarrow{DC} = \vec{a} \) and \( \overrightarrow{AD} = \overrightarrow{BC} = \vec{b} \).

The resulting figure is a parallelogram and the vector \( \vec{c} \) is the diagonal of that figure.

From the definition of equality given in the preceding chapter it is clear that this definition of addition is independent of the relative positions of the vectors in the plane.

The following laws hold under vector addition:
1. Commutative

\[ \overrightarrow{AD} + \overrightarrow{AB} = \overrightarrow{a} + \overrightarrow{b} \quad \overrightarrow{a} + \overrightarrow{b} = \overrightarrow{b} + \overrightarrow{a} \]

\[ \overrightarrow{AE} + \overrightarrow{AD} = \overrightarrow{b} + \overrightarrow{a} \]

2. Associative

\[ \overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{c}) = (\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c} \]

A verification of this law can be seen by constructing a polygon having the vectors \( \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \), as sides. Then from figure 5

\[(\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c} = \overrightarrow{PS} + \overrightarrow{SR} = \overrightarrow{QR},\]

\[\overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{PF} + \overrightarrow{PR} = \overrightarrow{QR}\]

3. Distributive

a) The product of a vector and the scalar sum can be distributed throughout the sum.

\[(m + n)\overrightarrow{a} = m\overrightarrow{a} + n\overrightarrow{a}\]

b) The product of a scalar by a vector sum can be distributed throughout that scalar sum

\[n(\overrightarrow{a} + \overrightarrow{b}) = n\overrightarrow{a} + n\overrightarrow{b}\]
The first assumption follows from the fact that both sides represent a vector with magnitude \((m + n)\vec{a}\) with the same or opposite direction of \(\vec{a}\) according as \((m + n)\) is positive or negative.

The second can be verified by considering the figure 6 which defines the vectors \(\vec{a}, \vec{b}, n\vec{a}, n\vec{b}\).

\[\text{FIGURE 6}\]

Since these two triangles are similar (all angles are equal since multiplication by a scalar does not affect the angles), the corresponding sides are proportional. This constant of proportionality is \(n\). Hence
\[n\vec{a} = \vec{b}.\]

9. \textbf{Subtraction of Vectors:} Subtraction of a vector \(\vec{b}\) from another
vector \( \overrightarrow{a} \) is defined as the process of adding to \( \overrightarrow{a} \) the vector which is equal in magnitude to \( \overrightarrow{b} \), but with opposite sign.

10. **Components of Vectors**: If we consider the vectors in the plane any vector \( \overrightarrow{v} \) can be expressed as the sum of two others.

   Let \( \overrightarrow{a}, \overrightarrow{b} \) be the generating vectors.
With any point $O$ as the origin take $\overrightarrow{OF} = \overrightarrow{r}$, and an $\overrightarrow{OP}$ as diagonal construct a parallelogram with sides $\overrightarrow{OA}$ parallel to $\overrightarrow{BF}$ and $\overrightarrow{OB}$ parallel to $\overrightarrow{AP}$. Then $\overrightarrow{r}$ is expressible as the sum

$$\overrightarrow{r} = \overrightarrow{OB} + \overrightarrow{PB} = \overrightarrow{OA} + \overrightarrow{AP}
= x\overrightarrow{a} + y\overrightarrow{b}.$$

Thus $\overrightarrow{r}$ is the resultant of the two vectors $x\overrightarrow{a}$, $y\overrightarrow{b}$ which are called the components of $\overrightarrow{r}$ in the given directions.

11. The Unit Vectors: An important resolution of vectors in plane geometry is that in which the two directions are mutually perpendicular. The unit vectors parallel to these axes will be denoted by $\overrightarrow{i}$, $\overrightarrow{j}$. The rotations about the axes $OX$, $OY$ are from $X$ to $Y$, from $Y$ to $-X$, from $-X$ to $-Y$, from $-Y$ to $X$ respectively.

If $X$, $Y$ are the lengths of $\overrightarrow{OA}$, $\overrightarrow{OB}$ respectively measured in these directions, the vector $\overrightarrow{OP}$ is

$$\overrightarrow{r} = X\overrightarrow{i} + Y\overrightarrow{j}.$$
If \( \alpha, \beta \) are the angles which \( \overline{OP} \) makes with the axes, then \( \cos \alpha \) and \( \cos \beta \) are the direction cosines of the line \( \overline{OP} \) and

\[
x = r \cos \alpha \quad y = r \cos \beta.
\]

It is clear from the figure that

\[
r^2 = x^2 + y^2.
\]

This gives the distance of \( P \) from the origin in terms of the coordinates.

12. **Scalar or Dot Product of Two Vectors:** The scalar or dot product is obtained by multiplying the cosine of the angle between the two vectors by the product of the magnitudes. It is symbolized by \( \overline{a} \cdot \overline{b} \). The result of scalar multiplication is a scalar.

\[
\overline{a} \cdot \overline{b} = (\text{mag } \overline{a})(\text{mag } \overline{b}) (\cos \alpha).
\]

Symbolically

\[
\overline{a} \cdot \overline{b} = |\overline{a}| |\overline{b}| \cos \alpha.
\]

An angle of \( 180^\circ \) indicates collinear vectors. Since the cosine of this angle is unity, the dot product is equal to the product of the numerical value of the magnitudes.

If the vectors are perpendicular the angle between them becomes \( \pm 90^\circ \).

The cosine vanishes and the dot product of the two vectors is zero.

This suggests a table for determining the scalar products of the two fundamental vectors mentioned previously.
\[ \vec{a} \cdot \vec{a} = b \cdot b = 1 \]
\[ \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = 0. \]

The scalar product obeys the same laws as the product of ordinary algebra.

11. **Vector Product of Two Vectors**: The vector product or cross product of two vectors \( \vec{a} \) and \( \vec{b} \) is the vector \( \vec{\times} \) which is normal to the plane of \( \vec{a} \) and \( \vec{b} \) on which rotation from \( \vec{a} \) to \( \vec{b} \) through an angle of less than 180° appears counter clockwise.

This vector has magnitude obtained by multiplying the magnitudes of \( \vec{a} \) and \( \vec{b} \) by the sine of the angle between the vectors.

![](image)

**FIGURE 10**

The result of the vector product is another vector

\[ \vec{r} = |\vec{a}| |\vec{b}| \sin \alpha. \]

In the case of parallel vectors the angle between them becomes 0° or
180°. The sine vanishes and the result is the zero vector.

If the two vectors are perpendicular the angle between them is ±90° and the sine is unity. Therefore, the magnitude of the resulting vector is equal to the product of the original vector.

If in vector multiplication the vectors \( \vec{a} \) and \( \vec{b} \) were changed the scalars would remain unchanged but the angle would be reversed and

\[
\vec{a} \times \vec{b} = -\vec{b} \times \vec{a},
\]

which suggests that vector multiplication is not commutative.

Vector multiplication is, however, distributive with respect to addition. But the associative law does not hold. That is

\[
\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}
\]

but

\[
\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}.
\]

In the last equation the left side represents a vector which is mutually perpendicular to \( \vec{b} \) and \( \vec{c} \). The right side results in a vector perpendicular to both \( \vec{a} \) and \( \vec{b} \). Unless the vectors are in the same plane, the right side of the equation cannot equal the left.
CHAPTER IV

ON THE APPLICATION OF VECTORS TO PLANE ANALYTIC GEOMETRY

Distance of a Point from a Line: To find the perpendicular distance of a point \( P \) from the straight line represented by the vector

\[
\vec{x} = \vec{a} + \lambda \vec{b}
\]

or

\[
\vec{x} - \vec{a} = \lambda \vec{b}, \text{ where } \vec{b} \text{ is a unit vector,}
\]

\[\text{FIGURE 11}\]

let \( P, A \) be points whose position vectors (vectors emanating at the origin with terminal points on \( P, A \)) are given by \( \vec{p}, \vec{a} \) respectively, and \( \overline{PN} \) the perpendicular from \( P \) to the given line. From the definition of scalar product

\[
(\vec{a} - \vec{p}) \cdot \vec{b} = |\vec{a} - \vec{p}| \cos \theta
\]

the perpendicular distance \( d \) is given by the equation

\[
(2) \quad d = |d| = \sqrt{(\vec{a} - \vec{p})^2 - (\vec{a} - \vec{p})^2 \cos^2 \theta}
\]

But

\[
(3) \quad \cos^2 \theta = \frac{(\vec{a} - \vec{p}) \cdot \vec{b}^2}{|\vec{a} - \vec{p}|^2}
\]
Also

\[ (4) \quad d = \| \overline{d} \| = \| \overline{a} - \overline{p} \| \sin \phi = \| \overline{a} - \overline{p} \| \sqrt{1 - \sin^2 \phi} \]

\[ = \| \overline{a} - \overline{p} \| \sqrt{1 - \frac{(\overline{a} - \overline{p}) \cdot \overline{b}}{\| \overline{a} - \overline{p} \|^2}} . \]

The last statements follow from the fact that \( \cos^2 \phi = 1 - \sin^2 \phi \). Then

\[ (5) \quad d = \sqrt{(\overline{a} - \overline{p})^2 - \left[ \overline{b} \cdot (\overline{a} - \overline{p}) \right]^2} . \]

To find the perpendicular distance from the line \( 4x - 3y + 15 = 0 \) to the point \( P(2,1) \).

Two points on the line are \( P_1(-3,1) \) and \( P_0(0,5) \). The unit vector \( \overline{b} \) is given by

\[ \overline{b} = \frac{3\overline{i} + 4\overline{j}}{\sqrt{9 + 16}} \]

\[ = \frac{3\overline{i} + 4\overline{j}}{5} . \]

Therefore

\[ d = \sqrt{(4 + 16) - \left[ \frac{6}{5} - \frac{16}{5} \right]^2} \]

\[ = \sqrt{\frac{400}{5}} \]

\[ = \frac{20}{5} \]

\[ = 4. \]
Division of a Line Segment in a Given Ratio: Consider the two points \( A \) and \( B \). We can find a third point \( C \) which divides the segment denoted by the vector \( \overrightarrow{AB} \) in the given ratio \( m \) to \( n \).

![Diagram of line segment with points A, B, and C](image)

It is clearly seen that if \( C \) lies between \( A \) and \( B \), then \( m/n \) is finite and positive. If \( C \) lies beyond \( B \) then \( m/n \) is finite and negative. If \( C \) lies beyond \( A \), then \( m/n \) is between \(-1\) and \( 0 \). We may represent the division of the points by

\[
(7) \quad \frac{\overrightarrow{AC}}{m} = \frac{\overrightarrow{BC}}{n}.
\]

If we denote the vector \( \overrightarrow{OA}, \overrightarrow{OC}, \overrightarrow{OB} \) by \( \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \) respectively then

\[
\overrightarrow{AC} = \overrightarrow{c} - \overrightarrow{a}
\]

\[
\overrightarrow{BC} = \overrightarrow{b} - \overrightarrow{c}.
\]

These can be written in the form

\[
n(\overrightarrow{c} - \overrightarrow{a}) = m(\overrightarrow{b} - \overrightarrow{c}).
\]

Solving this equation for \( \overrightarrow{c} \), we have

\[
(8) \quad \overrightarrow{c} = \frac{mb + na}{m + n}.
\]

To find the point \( P \) which divides \( P_1 (-1, -6) \) and \( P_2 (3, 0) \) so that \( P \) is four times as far from \( P_2 \) as from \( P_1 \) and \( C \) lies beyond \( P_1 \)
\[ \frac{P_1P}{PP_2} = \frac{1}{4}. \]

The coordinates of the points are
\[ x_1 = -1 \quad y_1 = -6 \]
\[ x_2 = 3 \quad y_2 = 0. \]

Then
\[ x = \frac{-1 - \frac{3}{4}}{1 - 1/4} = -2 \frac{1}{3} \]
\[ y = \frac{-6 \frac{3}{4}}{1 - 1/4} = -8. \]

The coordinates of the point are
\[ P(-2 \frac{1}{3}, -8). \]

The Distance Between Two Points: Suppose A and B are two given points. We can find the distance \( d \) between A and B in terms of the position-vectors \( \vec{a} \) and \( \vec{b} \) of A and B.

\[ d = |\vec{AB}| \]

But \( \vec{AB} = \vec{b} - \vec{a} \). Thus
\[ (9) \quad d^2 = |\vec{AB}|^2 = \vec{AB} \cdot \vec{AB} = (\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a}). \]
Therefore

(10) \( d = \sqrt{(b - a) \cdot (\overline{b} - \overline{a})} \).

To find the distance between the points \( P_1(1,3) \) and \( P_2(-5,5) \)

\[
d = \frac{|\overrightarrow{P_1P_2}|}{2}
\]

\[
\overrightarrow{P_1P_2} = \overrightarrow{P_1} - \overrightarrow{P_2}.
\]

Using the equation we get

\[
d = \sqrt{(1 + 5)^2 + (3 - 5)^2}
\]

\[
= \sqrt{40}.
\]

The **Area of a Triangle**: Given any three non-collinear points \( A, B, C \), we can find the area of the triangle in terms of the position-vectors \( \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \) of the points \( A, B, C \).

![Figure 15](image)

We begin by constructing the parallelogram of which \( \overrightarrow{AE}, \overrightarrow{AC} \) form two adjacent edges. Then the triangle determined by the points \( A, B, C \) is equal to one half the area of this parallelogram. The area of this parallelogram is \( |\overrightarrow{AB} \times \overrightarrow{AC}| \).
Therefore the area of the triangle is

\[ (11) \text{Area} = \frac{1}{2} \overrightarrow{AB} \times \overrightarrow{AC}. \]

Observe that

\[ (12) \overrightarrow{AB} = \overrightarrow{b} - \overrightarrow{c}, \quad \overrightarrow{AC} = \overrightarrow{c} - \overrightarrow{a}. \]

Hence

\[ (13) \overrightarrow{AB} \times \overrightarrow{AC} = (\overrightarrow{b} - \overrightarrow{a}) \times (\overrightarrow{c} - \overrightarrow{a}). \]

This reduces to

\[ \overrightarrow{AB} \times \overrightarrow{AC} = \overrightarrow{b} \times \overrightarrow{c} + \overrightarrow{c} \times \overrightarrow{a} + \overrightarrow{a} \times \overrightarrow{b}. \]

\[ \overrightarrow{AB} \times \overrightarrow{AC} \] is the magnitude of the vector given by the right side of the equation. Therefore the required area of the triangle is given by

\[ \frac{1}{2} \overrightarrow{AB} \times \overrightarrow{CD}. \]

Vector Equation of a Circle: Since a circle is the vector equidistance from a fixed point \( C \).

\[ (14) \quad \| \overrightarrow{x} - \overrightarrow{c} \| \]

represents the radius of the circle. So that

\[ (15) \quad (\overrightarrow{x} - \overrightarrow{c})^2 = \mathcal{R}^2 \]

is the vector equation of the circle with radius \( \mathcal{R} \) and center at \( C \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{circle_vector_equation.png}
\caption{Vector Equation of a Circle}
\end{figure}
Vector Equation of the Tangent to the Circle: The tangent drawn to the circle is perpendicular to the radius vector at the point of tangency. Let $P_0$ be the point of tangency. Then if $X$ is any point on the tangent line through this point we get

\[(16) \ (\vec{x} - \vec{P_0}) \cdot (\vec{P_0} - \vec{c}) = 0.\]

**FIGURE 17**

**Pencils of Circles:** Consider a circle with center $c_1$ and radius $\rho_1$.

Then from (15) we have

\[(17) \ x^2 - 2\vec{x} \cdot \vec{c}_1 + c_1^2 - \rho_1^2 = 0.\]

This equation represents the equation of a circle.
Now let us define a scalar \( J_i \) and a vector \( \vec{p} \), in such a manner that

\[
\begin{align*}
(18) \quad J_i &= \lambda_i (c_1^2 - p_1^2) \\
(19) \quad \vec{p}_1 &= -2 \lambda_i \vec{c}_1.
\end{align*}
\]

Hence

\[
(20) \quad \vec{c}_1 = \frac{\vec{p}_1}{2 \lambda_i}, \quad \vec{c}_1^2 = c_1^2 - \frac{J_i}{\lambda_i}.
\]

Now (17) takes the form

\[
(21) \quad \lambda_i \vec{x} + \vec{x} \cdot \vec{p}_1 + \gamma_i = 0.
\]

This equation enables us to consider the line circle, that is the circle from which \( \lambda_i = 0 \). It is clearly seen that this represents the equation of the circle of infinite radius.

To examine the nature of this line circle suppose the line intersects the line drawn from the origin toward \( \vec{c}_1 \) at some point \( Q_1 \). Then

\[
(22) \quad \ell_i = \| \vec{c}_1 - \vec{q}_1 \| \quad \ell_i = (\vec{c}_1 - \vec{q}_1)^2 = c_1^2 - \vec{c}_1 \cdot \vec{q}_1 + q_1^2.
\]

From (18) we have

\[
\gamma_i = \lambda_i (2 \vec{c}_1 \cdot \vec{q}_1 - q_1^2)
= 2 \lambda_i \vec{c}_1 \cdot \vec{q}_1 - \lambda_i q_1^2.
\]

Observe that

\[
\vec{p}_1 = 2 \lambda_i \vec{c}_1.
\]

If \( \lambda_i = 0 \), then

\[
\gamma_i = \vec{p}_1 \cdot \vec{q}_1, \quad \text{so that the equation of the circle from (21) becomes}
(23) \quad \vec{x} \cdot \vec{p}_1 - \vec{p}_1 - \vec{q}_1 = 0.
\]

or

\[
\vec{p}_1 \cdot (\vec{x} - \vec{q}_1) = 0.
\]

This represents the straight line perpendicular to \( \vec{p}_1 \) at \( Q_1 \).

Consider now a second circle
(24) \( \alpha_2 x^2 + \bar{y} \bar{c}_2 + y_2 = 0 \).

For arbitrary constants \( \alpha_1, \alpha_2 \) consider the equation

(25) \( \alpha_1 [\bar{y} \bar{x}^2 + \bar{z} \bar{y} + \bar{c}] + \alpha_2 (\bar{y} \bar{x}^2 + \bar{z} \bar{x} \bar{y} + 1) = 0 \).

This leads to

(26) \( (\alpha_1 + \alpha_2 \bar{x}) x^2 + (\alpha_1 \bar{y} + \alpha_2 \bar{x} \bar{y}) \bar{x} + (\alpha_1 \bar{c} + \alpha_2 \bar{x} \bar{c}) = 0 \).

If

\[ \alpha_1 + \alpha_2 \bar{x} \neq 0, \]

we have the equation of a circle whose center is at

(27) \( \bar{c} = \frac{\alpha_1 \bar{y} + \alpha_2 \bar{c}}{-2(\alpha_1 + \alpha_2 \bar{x})} \).

If

\[ \alpha_1 + \alpha_2 \bar{x} = 0, \]

then (26) represents the line circle.

Any circle derived from two circles is said to be a member of a pencil
of circles determined by those two circles.

There are several possible relations between circles. Some of them
are

(1) Two concentric circles.
(2) Two circles which do not touch. That is, one is inside the other.
(3) Two may intersect at one point.
(4) Two may intersect at two points.

Let us consider concentric circles.

From (20) and (27) we have that the center \( \bar{c} \) of the resulting circle is

given by

(28) \( \bar{c} = \frac{\alpha_1 \bar{y} + \alpha_2 \bar{c}}{\alpha_1 + \alpha_2 \bar{x}} \).

If \( \bar{c}_1 = \bar{c}_2 \), then

(29) \( \bar{c} = \frac{\alpha_1 \bar{y} + \alpha_2 \bar{c}}{\alpha_1 + \alpha_2 \bar{x}} \bar{c}_1 = \bar{c}_2 \)

If this relationship exists, the circles are concentric, so that all
circles of the pencil have the same center.
For non-concentric circles.

If $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are not equal then $\mathfrak{C}_1$ and $\mathfrak{C}_2$ determine a straight line. The equation of this line is given by

\[ \mathbf{c} = \mathbf{c}_1 + \lambda (\mathbf{c}_2 - \mathbf{c}_1) \]

where

\[ \mathbf{c} = (1 - \lambda) \mathbf{c}_1 + \lambda \mathbf{c}_2 \]

Since $\lambda$ is arbitrary, the only condition that a vector $\mathbf{x}$ to the point $X$ lie on a line $\mathbf{c}_1$ and $\mathbf{c}_2$ is that

\[ \mathbf{x} = \lambda \mathbf{c}_1 + \mathbf{m} \mathbf{c}_2 \]

then $\lambda + \mathbf{m} = 1$. Observe that equation (29) can be put in the form

\[ \mathbf{c} = \frac{K_1 \lambda_1}{K_1 \lambda_1 + K_2 \lambda_2} \mathbf{c}_1 + \frac{K_2 \lambda_2}{K_1 \lambda_1 + K_2 \lambda_2} \mathbf{c}_2 \]

This fulfills the condition that $\mathbf{c}$ lie on the same line as $\mathfrak{C}_1$ and $\mathfrak{C}_2$. Therefore the centers of the members of the pencil lie on a line through $\mathfrak{C}_1$ and $\mathfrak{C}_2$ (that is, centers of the given two circles). Moreover, the
position of the center of a member circle can be found in this way. From

(30) we get

\[ (33) \quad \overline{\theta} - \overline{\theta}_1 = \lambda (\overline{\theta}_2 - \overline{\theta}_1) \]
\[ \overline{\theta} - \overline{\theta}_2 = \mu (\overline{\theta}_1 - \overline{\theta}_2) \]

or

\[ \lambda = \frac{\overline{\theta} - \overline{\theta}_1}{\overline{\theta}_2 - \overline{\theta}_1} \quad \text{and} \quad \mu = \frac{\overline{\theta} - \overline{\theta}_2}{\overline{\theta}_1 - \overline{\theta}_2} = -\frac{\overline{\theta}}{\overline{\theta}_2 - \overline{\theta}_1} \]

\[ \overline{\theta} - \overline{\theta}_1 = \frac{\mu \lambda + K}{\mu \lambda + \lambda} \]

Then

\[ (34) \quad \frac{\overline{\theta} - \overline{\theta}_1}{\overline{\theta}_1 - \overline{\theta}_2} = \frac{\mu \lambda}{\mu \lambda + \lambda} \]

Finally

\[ (35) \quad \frac{\overline{\theta} - \overline{\theta}_1}{\overline{\theta}_2 - \overline{\theta}} = \frac{\mu \lambda}{\mu \lambda + \lambda} \]

Hence as \( \frac{\mu \lambda}{\mu \lambda + \lambda} \) varies we get a family of circles with centers on the line of \( \overline{\theta}_1 \) and \( \overline{\theta}_2 \). For the members for which \( \mu \lambda + \lambda = 0 \) we have a line circle perpendicular to \( K, \overline{\theta}_1 + K \overline{\theta}_2 \).

\[ (36) \quad \lambda x^2 + \overline{\theta}_1 \cdot x + K = 0. \]

If in \( K, \overline{\theta}_1 + K \overline{\theta}_2 \) we replace \( \overline{\theta}_1 \) and \( \overline{\theta}_2 \) by their equals we get

\[ (37) \quad K, \overline{\theta}_1 + K \overline{\theta}_2 = K, (-2 \lambda \overline{\theta}_1) \times K \overline{\theta}_2 (-2 \lambda \overline{\theta}_2 \overline{\theta}_1) = \]
\[ = -2 \lambda \overline{\theta}_1 \overline{\theta}_1 = 2 \lambda \overline{\theta}_2 \overline{\theta}_1 \overline{\theta}_2 \]

Therefore we have the line circle perpendicular to the line through \( \overline{\theta}_1 \) and \( \overline{\theta}_2 \).

To consider contact points recall that
(38) \( \lambda_1 x^2 + \overline{p}_1 \cdot \overline{x} + \overline{r}_1 = 0 \)
(39) \( \lambda_2 x^2 + \overline{p}_2 \cdot \overline{x} + \overline{r}_2 = 0 \).

General equation for a member of the family is

(39) \( \lambda_1 (\lambda_2 x^2 + \overline{p}_1 \cdot \overline{x} + \overline{r}_1) + \lambda_2 (\lambda_2 x^2 + \overline{p}_2 \cdot \overline{x} + \overline{r}_2) = 0 \)

for \( \lambda_1 \neq 0, \lambda_2 \neq 0 \).

If the two given circles have a point in common then every member of the family has that point in common.

If there is no common point the family has no point in common.

**Power of a Point and Radical Axis:**

Definition 1: Let \( P_0 \) be a point not on the circle (1). Then

(40) \((\overline{P}_0 - \overline{a})^2 - \overline{c}^2 \neq 0\)

is called the power of \( P_0 \).

---

**FIGURE 19**

Definition 2: The line circle of any pencil is called the radical axis of
the pencil, since
\[ \mathcal{K}_1 \mathcal{K}_1 + \mathcal{K}_2 \mathcal{K}_2 = 0 \]
and since from the equation
\[ \mathcal{K}_1 \mathcal{K}_1 + \mathcal{K}_2 \mathcal{K}_2 = 2 \mathcal{K}_1 \mathcal{K}_2 \mathcal{K}_2 \mathcal{K}_2 = 0 \]
the radical axis becomes
\[ \mathcal{K}_1 \mathcal{K}_1 (\overline{c_2} - \overline{c_1}) \mathcal{K} + \mathcal{K}_2 \mathcal{K}_2 (\overline{c_1} - \overline{c_2}) \mathcal{K} = 0 \]
This reduces to
\[ 2(\overline{c_2} - \overline{c_1}) \mathcal{K} + \mathcal{K}_2 \mathcal{K}_2 (\overline{c_1} - \overline{c_2}) \mathcal{K} = 0 \]
The equation
\[ x^2 - 2\overline{c_1} \cdot \mathcal{K} + c_1^2 - \mathcal{E}_1^2 = x^2 - 2\overline{c_2} \cdot \mathcal{K} + c_2^2 - \mathcal{E}_2^2 \]
can be written in the form
\[ (\mathcal{K} - \overline{c_1})^2 - \mathcal{E}_1^2 = (\mathcal{K} - \overline{c_2})^2 - \mathcal{E}_2^2 \]
Hence, the power of any point on the radical axis with respect to any two
given circles of the pencil except the line circle, are equal.

For the orthogonal circles consider the two circles
\[ x^2 - 2\overline{c_1} \cdot x + a_1 = 0 \]
\[ x^2 - 2\overline{c_2} \cdot x + a_2 = 0 \]
\[ a_1 = c^2 - \mathcal{E}_1^2 \]. Let \( Q \) be a point of intersection of these two cir-
cles.

If these circles are orthogonal then \((\overline{c_1} - \overline{q}) \cdot (\overline{c_2} - \overline{q}) = 0\)

\[ \text{Figure 20} \]
Adding the equation of (44) we get
\[ 2x^2 - 2(\bar{c}_1 + \bar{c}_2) \cdot \bar{x} + \omega_i + \omega_2 = 0. \]

But for \( \bar{x} = \bar{q} \) we have

(45) \[ 2q^2 - 2(\bar{c}_1 - \bar{c}_2) \cdot \bar{q} + \omega_i + \omega_2 = 0. \]

(46) \[ (\bar{c}_1 - \bar{q}) \cdot (\bar{c}_2 - \bar{q}) = q^2 - (\bar{c}_1 + \bar{c}_2)q + \bar{c}_1 \cdot \bar{c}_2 = 0. \]

Multiplying (46) by 2 and subtracting (45) from this the result is

(47) \[ 2\bar{c}_1 \cdot \bar{c}_2 - \omega_i - \omega_2 = 0. \]

which is the condition for orthogonality we seek.

Hence we see that if the circles are orthogonal at \( Q_1 \) then they are orthogonal at \( Q_2 \).

**A Circle Orthogonal to a Family:**

**General equation:**

Let the circle

(48) \[ x^2 - 2\bar{x} \cdot \bar{c} + \omega = 0 \]

be orthogonal to the circles

(49) \[ x^2 - 2\bar{x} \cdot \bar{c}_1 + \omega_i = 0 \]
\[ x^2 - 2\bar{x} \cdot \bar{c}_2 + \omega_2 = 0. \]

Any member of the pencil determined by (49) can be written

(50) \[ (K_1 + K_2) x^2 - 2(K_1 \bar{c}_1 - K_2 \bar{c}_2) \cdot \bar{x} + K_1 \omega_i + K_2 \omega_2 = 0. \]

To determine the orthogonality of (48) to (50)

\[ 2 \bar{c} \cdot \frac{K_1 \bar{c}_1 + K_2 \bar{c}_2}{K_1 + K_2} - \omega = \frac{K_1 \omega_i + K_2 \omega_2}{K_1 + K_2} = \]
\[ = \frac{K_1}{K_1 + K_2} (\omega + \omega_i) + \frac{K_2}{K_1 + K_2} (\omega + \omega_2) - \omega = \frac{K_1 \omega_i + K_2 \omega_2}{K_1 + K_2} = 0, \]

for \( 2 \bar{c} \cdot \bar{c}_1 - \omega - \omega_i = 0 \)
\[ 2 \bar{c} \cdot \bar{c}_2 - \omega - \omega_2 = 0. \]
Therefore the circle (48) is orthogonal to the family determined by (49).

Claim that the center of (48) lies on a radical axis of the pencil. To see this put in equation (42) $\Omega$ in place of $x$. We then get

$$2(\overline{c_2} - \overline{c_1}) \cdot \overline{c} + \omega, - \omega = 2 \overline{c_2} - \overline{c} - \omega - \omega = (2 \overline{c_1} \cdot \overline{c} - \omega - \omega) = 0.$$ 

If two different circles are orthogonal to a pencil of circles then obviously any one member of the pencil determined by the two different circles. Hence the two pencils are orthogonal.
BIBLIOGRAPHY

