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Some applications of vector methods to plane geometry and plane trigonometry

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SOME APPLICATIONS OF VECTOR METHODS
TO
PLANE GEOMETRY AND PLANE TRIGONOMETRY

A THESIS
SUBMITTED TO THE FACULTY OF ATLANTA UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF MASTER OF SCIENCE

BY
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ATLANTA, GEORGIA
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ACKNOWLEDGMENT

During the preparation of this thesis, I have been greatly indebted to Mr. Chas. H. Pugh who was my instructor in Vector Analysis and who made valuable suggestions which have been incorporated in the work. It was his manner of treating Vector Analysis that aroused my interest in the application of its methods to plane geometry and plane trigonometry.

My thanks are also due to Mr. C. B. Dansby who read the proof sheets and made many excellent suggestions which I was glad to adopt.
INTRODUCTION

In 1832, Bellavitis devised the Calcoladelle Equipollenze which actually dealt systematically with the geometric addition of vectors and the equality of vectors. The systems of W.R. Hamilton and H.G. Grassman established 1835-44 may be regarded as the parents of Vector Analysis. The two authors worked independently and along different lines—the former on the subject of quaternion, a sort of "sum" or complex of a scalar and a vector, though originally defined as the "quotient" of two vectors, and the latter on algebra of geometric forms. These devotees strove faithfully to prove its power and usefulness in other branches of mathematics. However, neither of these systems met the needs of physicists or applied mathematicians until simplified by Heaviside and W. Gibbs, the former in England and the latter in America.

The present century has witnessed the appearance of an Italian school of vector analysts represented by R. Marcolongo and C. Burali-Forti, both of whom have been chiefly influenced by Grassman and Hamilton.¹

This study by no means claims to treat of the geometry of the line and plane to which vector methods may be applied but to cite and explain its power and utility in regard to some theorems and formulas which may be proved or verified


by means of vector algebra and vector geometry. Several of these cases display obvious advantages. However, vector analysis renders its greatest service in the domains of mechanics and mathematical physics.

The discussion and explanation of principles and ideas to be used are followed by their applications to the proof of familiar theorems from plane geometry and the verification of familiar and fundamental formulas from plane trigonometry. Both systems have been developed by means of a logical organization of the material. The laws of vector algebra and vector geometry have been carefully applied lest from analogy to the scalar quantity some properties may be attributed to the mathematical vector which do not apply to it.

Care has also been taken in drawing and placing the figures so that they fall directly under the eye in immediate connection with the text.
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CHAPTER I

FUNDAMENTAL PRINCIPLES AND OPERATIONS

The mathematician and the physicist deal with quantities which they find convenient to classify as scalar and vector quantities. Scalar quantities can be conveniently represented when the unit of scale is fixed.¹ Some familiar examples of scalars are mass, volume, density, time, length, and temperature. On the other hand no scale of numbers can represent adequately the other kind of quantity. The scale of numbers would only represent their magnitude but not their direction.² These quantities are known as vectors. Some familiar examples of which are velocity, weight, acceleration, force, potential, displacement, and momentum.

Vector analysis which treats of these quantities has three divisions:

1. Vector Algebra
2. Vector Geometry
3. Vector Calculus

In order to reveal the power and utility of vector methods in plane geometry and plane trigonometry, we shall have need for vector algebra and vector geometry.

A scalar quantity, or briefly a scalar, has magnitude but is not related to any direction in space.³ An example: 50 feet.

²Ibid.
³C.E. Weatherburn, op. cit., p. 1.
A vector quantity, or briefly a vector, has magnitude and is definitely related to a direction in space. An example: 50 feet northward.

Vectors having the same magnitude and the same sense or direction are equal\(^1\), i.e., if

\[
\overrightarrow{a} = \overrightarrow{b}
\]

\[
/\overrightarrow{a}/ = /\overrightarrow{b}/
\]

The module of a vector is the positive number which is the measure of its length. A \textit{unit vector}, then, is one whose magnitude or module is unity. A \textit{zero vector} or null vector is one whose module is zero. One vector is said to be the negative of another if it has the same magnitude but opposite direction.

\[
\begin{array}{c}
\overrightarrow{a} \\
\overrightarrow{-a}
\end{array}
\]

Fig. 1

In Fig. 1, note the geometric or graphic representation of a vector. The arrow is called a stroke. Its tail or its initial point is its origin, and its head or final point, its terminus.

Since vectors having the same magnitude and direction are equal or since two directed couples of points which can be transformed into each other by a parallel transformation define the same vector\(^2\), the "parallelogram law" enables us

\(^1\) The notations \(/a/\) and mod \(\overrightarrow{a}\) are used for module of \(a\).
Hilda Geiringer, \textit{op. cit.}, p. 1.
to see that a vector may be the sum of two or more component parts.

If two vectors \( \vec{a} \) and \( \vec{b} \) be drawn from a reference point 0 we see that the sum of the two vectors is the vector determined by the diagonal of the parallelogram (fig. 2).

That is, the sum or resultant of two vectors is found by placing the origin of the second upon the terminus of the first and drawing the vector from the origin of the first to the terminus of the second and the order in which they are added does not affect the sum\(^1\). Also \((\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \quad ---(1)\)

for in figure 3, \( \vec{a} = (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \).

The subtraction of one vector from another may be understood as the addition of its negative (fig. 4).

---

\(^{1}\) Hilda Geiringer, op. cit., p.9.
The figure shows that
\[ \vec{a} - \vec{b} = \vec{a} + (-\vec{b}) = \vec{c} \] \hspace{1cm} (2)

A vector is said to be multiplied by a positive scalar when its magnitude is multiplied by that scalar and its direction is left unaltered\(^1\). Thus if \( \vec{v} \) be a velocity of 10 knots South by West, \( 3 \frac{1}{2} \) times \( \vec{v} \) equals 35 knots with the direction still South by West. And

\[ m (n \vec{a}) = n (m \vec{a}) = (m n) \vec{a} \] \hspace{1cm} (3)

\[ (m + n)\vec{a} = m \vec{a} + n \vec{a} \] \hspace{1cm} (4)

\[ m(\vec{a} + \vec{b}) = m \vec{a} + m \vec{b} \] \hspace{1cm} (5)

\[ -(\vec{a} + \vec{b}) = -\vec{a} - \vec{b} \] \hspace{1cm} (6)

Thus the laws which govern addition, subtraction, and scalar multiplication of vectors are identical with those governing these operations in ordinary algebra\(^2\).

Vectors are said to be collinear when parallel to the same line; vectors which lie in or are parallel to the same plane are coplanar.

A vector can conveniently be divided into two components at right angles if vectors considered are coplanar; three if non-coplanar\(^3\).

---

\(^1\) Josiah Willard Gibbs, \textit{op. cit.}, p. 7.

\(^2\) Ibid., pp. 12 - 13. Also see equations (1), (3) - (6).

If vectors are coplanar, unit vectors along x and y axes are \( \hat{T} \) and \( \hat{J} \) respectively. In \( x\hat{T} \) and \( y\hat{J} \), \( x \) and \( y \) are the magnitudes or scalar parts of these vectors respectively and \( \hat{T} \) and \( \hat{J} \) the directions.

If vectors are non-coplanar, they may simply make up a rectangular system familiar in Solid Cartesian Geometry.

Similarly in \( z\hat{E} \), \( z \) is the magnitude and \( \hat{E} \) the direction (fig. 5).

In case two equal vectors are expressed in terms of one vector, or two non-coplanar vectors, or three non-coplanar vectors, the corresponding coefficients are equal\(^2\).

The scalar or dot product of two vectors \( \vec{a} \) and \( \vec{b} \) obeys the laws of ordinary multiplication and is a scalar defined

\[
\vec{a} \cdot \vec{b} = ab \cos(\vec{a}, \vec{b}) \tag{7}
\]

If \( \vec{a} \cdot \vec{b} = 0 \), then \( \vec{a} \perp \vec{b} \) \tag{8}

since \( \cos 90^\circ = 0 \), and \( \cos 270^\circ = 0 \).

\(^1\)Josiah Willard Gibbs, op. cit., pp. 18-21.

\(^2\)Ibid., pp. 17-18.
In figure 5,
\[
\begin{align*}
\mathbf{i} \cdot \mathbf{i} &= \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \\
\mathbf{i} \cdot \mathbf{j} &= \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0.
\end{align*}
\]

The vector or cross product of two vectors \( \mathbf{a} \) and \( \mathbf{b} \) is a vector quantity and is defined
\[
\mathbf{a} \times \mathbf{b} = \mathbf{\hat{e}} \, ab \sin(\mathbf{a}, \mathbf{b})
\]
where \( \mathbf{\hat{e}} \) is a unit vector perpendicular to the plane of \( \mathbf{a} \) and \( \mathbf{b} \) and \( ab \) in \( (\mathbf{a}, \mathbf{b}) \) the magnitudes of \( \mathbf{a} \) and \( \mathbf{b} \).

The commutative law does not hold for
\[
\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.
\]
The distributive law holds for
\[
(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}.
\]

If
\[
\mathbf{a} \times \mathbf{b} = 0,
\]
then
\[
\mathbf{a} \parallel \mathbf{b}
\]
since
\[
\sin 0^\circ = 0, \\
\sin 180^\circ = 0, \quad \sin 360^\circ = 0.
\]

In figure 5,
\[
\begin{align*}
\mathbf{i} \times \mathbf{i} &= \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \\
\mathbf{i} \times \mathbf{j} &= -\mathbf{j} \times \mathbf{i} = \mathbf{k} \\
\mathbf{j} \times \mathbf{k} &= -\mathbf{k} \times \mathbf{j} = \mathbf{i} \\
\mathbf{k} \times \mathbf{i} &= -\mathbf{i} \times \mathbf{k} = \mathbf{j}
\end{align*}
\]

\(^1\)Joseph George Coffin, op. cit., p. 35.
Summary of Chapter I. Scalar quantities possess magnitude only while vector quantities possess magnitude and direction. Equal vectors have equal magnitude and the same direction. A null of zero vector has module of zero. A vector is unaltered by translating it parallel to itself. Vectors are added according to the parallelogram law. To subtract one vector from another reverse its direction and add. Addition, subtraction and multiplication of vectors by a scalar follow the same laws as in algebra.

A vector may be resolved into three component parts parallel to any three non-coplanar vectors in the following manner:

\[ \vec{a} = x \vec{I} + y \vec{J} + z \vec{k} \]

where \( \vec{I}, \vec{J} \) and \( \vec{K} \) are unit vectors forming a left-handed rectangular system from Solid Cartesian Geometry.

The scalar product of two vectors is equal to the product of their length multiplied by the cosine of the angle between them, i.e.,

\[ \vec{a} \cdot \vec{b} = ab \cos(\vec{a}, \vec{b}). \]

If this product is zero, the vectors are perpendicular. The scalar products of \( \vec{I}, \vec{J}, \vec{K} \) are

\[ \vec{I} \cdot \vec{I} = \vec{J} \cdot \vec{J} = \vec{K} \cdot \vec{K} = 1, \]
\[ \vec{I} \cdot \vec{J} = \vec{J} \cdot \vec{K} = \vec{K} \cdot \vec{I} = 0. \]

The vector product of two vectors is equal in magnitude to the product of their lengths multiplied by the sine of the angle between them and the direction is the unit vector \( \vec{C} \),
normal to the plane of the two vectors, i.e.,

\[ \mathbf{a} \times \mathbf{b} = \mathbf{c} \quad ab \quad \sin \: (\mathbf{a}, \mathbf{b}). \]

If this product is zero, the two vectors are parallel. The commutative law does not hold. The vector products of \( \mathbf{i} \), \( \mathbf{j} \) and \( \mathbf{k} \) are

\[
\begin{align*}
\mathbf{i} \times \mathbf{i} &= \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \\
\mathbf{i} \times \mathbf{j} &= -\mathbf{j} \times \mathbf{i} = \mathbf{k} \\
\mathbf{j} \times \mathbf{k} &= -\mathbf{k} \times \mathbf{j} = \mathbf{i} \\
\mathbf{k} \times \mathbf{i} &= -\mathbf{i} \times \mathbf{k} = \mathbf{j}
\end{align*}
\]
CHAPTER II

VECTOR METHODS IN PLANE GEOMETRY

Frequently, problems in plane geometry can be easily and sometimes advantageously solved by vector methods; in this chapter we shall consider several forms of vector equations of a circle, some examples from plane geometry in this connection and still other examples concerning parallelograms and the triangle.

The vector equation of a circle may take on either of three forms according as the reference point is at center, on the circle or outside the circle\(^1\).

Let circle 0 be the circle with radius \(a\) and reference point at 0

\[
\text{Fig. 6}
\]

with \(\mathbf{r}\) tracing out the circle (fig. 6), then

\[
\mathbf{r} \cdot \mathbf{r} = a^2,
\]

\[
\mathbf{r}^2 = a^2.
\]

Or,

\[
\mathbf{r}^2 - a^2 = 0
\]

which is one form of the vector equation of a circle.

\(^1\)Joseph George Coffin, op. cit., pp. 61-63.
When reference point is outside the circle at 0, and \( \overline{r} \) and \( \overline{c} \) the fundamental vectors to a point on the circle and to the center respectively (fig. 7), then

\[
(\overline{r} - \overline{c})^2 = \overline{a}^2 \\
\overline{r}^2 - 2\overline{r} \cdot \overline{c} = \overline{a}^2 - \overline{c}^2
\]

a second form of the vector equation of a circle.

If the reference point 0 is on the circle (fig. 8),

then \( \overline{c} = \overline{a} \) and (14) becomes

\[
\overline{r}^2 - 2\overline{r} \cdot \overline{a} = 0.
\]

------------------------(15)
In connection with a theorem which follows, the vector equation of the bisector of the angle between two straight lines and the centroid of two points are needed.

To derive this equation let OA and OB be straight lines, parallel to unit vectors, $\vec{a}$ and $\vec{b}$ (figure 9). Take O as origin

$PQ$ is $\parallel$ OA, and let $P$ be any point on the bisector. Since $\overline{1} \parallel \overline{2}$ and then $\overline{2} \parallel \overline{4}$ so that $OQ = PQ$, and $OQ$ and $PQ$ are parallel to $\vec{b}$ and $\vec{a}$ respectively. Therefore, the position vector of $P$ is

$$ \vec{r} = t (\vec{a} + \vec{b}) \quad \text{(16)} $$

where $t$ is a variable scalar.

The vector equation of the bisector of the supplementary angle would be

$$ \vec{r} = t \left[ \vec{a} + (-\vec{b}) \right] = t(\vec{a} - \vec{b}) \quad \text{(17)} $$

since the direction of the unit vector is opposite that of $\vec{b}$.

The position vector of $P$ for the origin $O$ is $\overrightarrow{OP}$.

Given $n$ points whose position vectors relative to an origin $O$ are $\vec{a}_1, \vec{b}_1, \ldots$. The point $P$ whose position vector is

$$ \overrightarrow{OP} = \frac{p \vec{a} + q \vec{b} + \ldots}{p + q + \ldots} \quad \text{---------(18)} $$

if $p, q, \ldots$ are real numbers, is called the centroid or center of mean position.
The centroid of two points $A$, $B$ with associated numbers $p$, $q$ divides the line $AB$ in the ratio $q:p$ for then

$$\overline{OP} = \frac{p \overline{A} + q \overline{B}}{p + q}$$

whether $p$ and $q$ are negative or positive.\(^1\)

\(^1\)C. E. Weatherburn, \textit{op. cit.}, pp. 9 - 10.
Theorem 1. An angle inscribed in a semi-circle is measured by one half the intercepted arc.

In figure 8:
Given: \(a, \angle, b\) is inscribed.
To prove: \(a\) is measured by \(\frac{1}{2} \angle b\).

\[\overline{a} = 2\overline{b} = \overline{b}\]

Multiplying by \(\overline{a}\), we get \(\overline{a} \cdot (\overline{a} - 2\overline{b}) = \overline{a} \cdot b\).

By (15), \(\overline{a}^2 - 2\overline{a} \cdot \overline{b} = 0\).

\[\overline{a} \cdot b = 0, \text{ and } \angle (b, \overline{a}) = 90^\circ\]

But \(\angle b = 180^\circ\),
\[\angle (b, \overline{a}) = \frac{1}{2} \angle b\]

Theorem 2. The line of centers of two intersecting circles is perpendicular to the common chord (fig. 10).

Given: \(c_1\) and \(c_2\) intersecting at \(0\) and \(0'\);
\(c_1c_2\) is line of centers, \(00'\) is common chord.

To prove: \(00' \perp c_1c_2\).

From (15), equation of \(c_1\) : \(\overline{a}^2 - 2\overline{a} \cdot \overline{a}_1 = 0\).
Similarly equation of \(c_2\) : \(\overline{a}^2 - 2\overline{a} \cdot \overline{a}_2 = 0\).
Solving simultaneously, \(2\overline{a} \cdot \overline{a}_2 - 2\overline{a} \cdot \overline{a}_1 = 0\),
\[\overline{a} \cdot (\overline{a}_2 - \overline{a}_1) = 0\]
\[\overline{a} \cdot (c_2 - c_1) = 0\]
\[00' \cdot (c_2 - c_1) = 0\]
and \(00' \perp c_2c_1\).
Corollary. The two lines of centers, $O_1 O_3$ and $O_2 O_4$, of four equivalent circles, each tangent to the preceding one and the last tangent to the first, are perpendicular (fig. 10').

Fig. 10'

Theorem 3. Both pairs of opposite sides of a parallelogram are equal (fig. 11).

Fig. 11

Given: $\square ABCD$.

To prove: $AB = CD$ and $AC = BD$.

Const.: Join $A$ to $D$ and let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BD} = \vec{d}$, $\overrightarrow{AD} = \vec{e}$, $\overrightarrow{CD} = \vec{b}$, and $\overrightarrow{AC} = \vec{c}$.

Then

$\vec{a} + \vec{d} = \vec{e}$ and $\vec{c} + \vec{b} = \vec{e}$.

\[\therefore \vec{a} + \vec{d} = \vec{c} + \vec{b}\]

and

$\vec{a} - \vec{b} = \vec{c} - \vec{d}$.

\[\therefore \text{by (13), we have}\]

\[
(\vec{a} - \vec{b}) \times (\vec{c} - \vec{d}) = 0,
\]

\[
\vec{a} \times \vec{c} - \vec{b} \times \vec{c} - \vec{a} \times \vec{d} - \vec{b} \times \vec{d} = 0.
\]
\[ \overline{e} \text{ ac } \sin A - \overline{e} \text{ bc } \sin A - \overline{e} \text{ ad } \sin A + \overline{e} \text{ bd } \sin A = 0, \]
\[ \overline{e} \left[ \sin A (ac + bd - ad - bc) \right] = 0, \]
\[ \overline{e} \left\{ \sin A \left[ c(a - b) - d(a - b) \right] \right\} = 0, \]
and
\[ \overline{e} \sin A (c - d)(a - b) = 0. \]

Since \( \overline{e} \neq 0 \) and \( \sin A \neq 0 \) in this case\(^1\), then \( a = b \) or \( c = d \).

If \( a = b \), then \( c = d \); or if \( c = d \), then \( a = b \) for \( a - b = c - d \).

\[ \therefore \quad a = b \text{ or } AB = CD \quad \text{and} \quad c = d \text{ or } AC = BD. \]

**Theorem 4.** In a parallelogram the diagonal divides it into two congruent triangles. (See fig. 11.)

**Given:** \( \square ABCD \).

**To prove:** \( \triangle ABD \cong \triangle ACD \).

From theorem 3, \( AB = CD \) and \( AC = BD \) and by identity, \( AD = AD \).

\[ \therefore \quad \triangle ABD \cong \triangle ACD. \]

**Corollary.** In a parallelogram, opposite sides and opposite angles are equal.

**Theorem 5.** In a parallelogram the two diagonals bisect each other (fig. 12).

![Fig. 12](image)

**Given:** \( \square A B C D \) with diagonals \( AD \) and \( BC \).

\(^1\) \( \sin A \neq 0 \) for if \( \sin A = 0 \), \( \angle A \) would equal \( 0^\circ, 180^\circ \) or \( 360^\circ \) and the figure would not be a parallelogram.
Let \( \overline{AB} = \overline{a}, \overline{BD} = \overline{d}, \overline{OD} = \overline{b}, \overline{AC} = \overline{c}, \overline{CO} = \overline{f}, \overline{OB} = \overline{g}, \overline{AO} = \overline{h}, \) and \( \overline{OD} = \overline{I}. \)

Then
\[
\overline{f} + \overline{I} = \overline{b}
\]
and
\[
\overline{h} + \overline{g} = \overline{a},
\]
\[
\overline{f} + \overline{I} = \overline{h} + \overline{g}
\]
and
\[
\overline{f} - \overline{g} = \overline{h} - \overline{I}.
\]

By (13), we have
\[
(\overline{f} - \overline{g}) \times (\overline{h} - \overline{I}) = 0,
\]
Then
\[
\overline{f} \times \overline{b} - \overline{g} \times \overline{h} - \overline{f} \times \overline{I} + \overline{g} \times \overline{I} = 0,
\]
\[
\overline{e} f h \sin (f, h) - \overline{e} g h \sin (f, h)
\]
\[
- \overline{e} f l \sin (f, h) + \overline{e} g l \sin (f, h) = 0,
\]
\[
\overline{e} [\sin (f, h) (fh - gh - fl + gl)] = 0,
\]
and
\[
\overline{e} \{\sin (f, h) [h(f - g) - l(f - g)]\} = 0,
\]
or
\[
\overline{e} \sin (f, h)(h - 1)(f - g) = 0.
\]

Since \( \overline{e} \neq 0 \) and \( \sin (f, h) \neq 0 \) in this case, then
\[
h = 1
\]
or
\[
f = g.
\]
If \( h = 1 \) then \( f = g \)
or if \( f = g \) then \( h = 1 \)
for \( f - g = h - l \),
\[
\therefore f = g \text{ or } \overline{CO} = \overline{OB}
\]
and
\[
h = 1 \text{ or } \overline{AO} = \overline{OD}.
\]

\(^1\) \( \sin (f, h) \neq 0 \) for if \( \sin (f, h) = 0, \angle (f, h) \) would equal \( 0^\circ, 180^\circ, \) or \( 360^\circ \) and the figure would not be a parallelogram.
Theorem 6. The line which joins one vertex of a parallelogram to the midpoint of an opposite side trisects the diagonal from an adjacent vertex (fig. 13).

![Fig. 13](image)

Given: \( \triangle O B C D, E \) the midpoint of \( O D, O C \) the diagonal, and \( B E \) meeting \( OC \) at \( R \).

To prove: \( OR = 1/3 OC \).

Let \( \overline{OD} = \overline{a} \) and \( \overline{OE} = \overline{b} \).

Then prove

\[
\overline{OR} = 1/3 (\overline{a} + \overline{b}),
\]

\[
\overline{OR} = \overline{OE} + \overline{ER} = 1/2 \overline{a} + x(\overline{b} - 1/2 \overline{a})
\]

where \( x \) is an unknown scalar representing some portion or fractional part of \( EB \) and

\[
\overline{OR} = y (\overline{a} + \overline{b})
\]

where \( y \) is an unknown scalar representing some fractional part of \( OC \) to be shown equal to 1/3.

Hence

\[
1/2 \overline{a} + x(\overline{b} - 1/2 \overline{a}) = y (\overline{a} + \overline{b})
\]

or

\[
1/2 \overline{a} + x \overline{b} - 1/2 x \overline{a} = y \overline{a} + y \overline{b}
\]

or

\[
x \overline{b} + 1/2(1 - x)\overline{a} = y \overline{a} + y \overline{b}.
\]

By equating corresponding coefficients (p. 5), we get

\[
x = y \quad \text{and} \quad 1/2 (1 - x) = y.
\]

Solving simultaneously, we obtain

\[
y = 1/3.
\]
Corollary. If $OE = \frac{1}{n} OD$, then

$$OR = \frac{1}{n + 1} OC.$$

Theorem 7. The sum of the squares of the diagonals of a parallelogram is equal to twice the sum of the squares of two consecutive sides (figure 14).

![Diagram of a parallelogram with diagonals and sides labeled]

Fig. 14

Given: $\square ABCD$ with diagonals $BD$ and $AC$.

To prove: $(AC)^2 + (BD)^2 = 2 \left[ (AD)^2 + (AB)^2 \right]$.

With $B$ as origin, let $\overline{AB} = \overline{a}$, $\overline{AD} = \overline{b}$, $\overline{BD} = \overline{c}$, and $\overline{AC} = \overline{d}$.

Proof:

$$\overline{c} = \overline{a} + \overline{b},$$

and

$$\overline{d} = \overline{a} + \overline{b}.$$  

Then

$$\overline{c} \cdot \overline{c} = (\overline{a} + \overline{b}) \cdot (\overline{a} + \overline{b}) = \overline{a} \cdot \overline{a} + 2 \overline{a} \cdot \overline{b} + \overline{b} \cdot \overline{b},$$

and

$$\overline{d} \cdot \overline{d} = (\overline{a} - \overline{b}) \cdot (\overline{a} - \overline{b}) = \overline{a} \cdot \overline{a} - 2 \overline{a} \cdot \overline{b} + \overline{b} \cdot \overline{b}.$$

Adding, we get

$$\overline{c} \cdot \overline{c} + \overline{d} \cdot \overline{d} = 2 (\overline{a} \cdot \overline{a} + \overline{b} \cdot \overline{b})$$

or

$$(AC)^2 + (BD)^2 = 2 \left[ (AD)^2 + (AB)^2 \right].$$
Theorem 9. The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides (figure 15).

![Figure 15](image.png)

**Fig. 15**

Given: \( \triangle ABC \), \( CM \) bisecting \( \angle C \).

To prove: \( AC : BC = AM : BM \).

With \( C \) as origin, let \( \overrightarrow{CA} = \overrightarrow{a} \) and \( \overrightarrow{CB} = \overrightarrow{b} \).

Then the vector equation of line \( CM \) is by (16),

\[
\overrightarrow{r} = t \left( \frac{\overrightarrow{b}}{b} + \frac{\overrightarrow{a}}{a} \right)
= t \frac{a \overrightarrow{b} + b \overrightarrow{a}}{ba}
\]

where \( t \) is a variable scalar. Let \( t = \frac{b \overrightarrow{a}}{b + a} \), then

\[
\overrightarrow{r} = \frac{a \overrightarrow{b} + b \overrightarrow{a}}{b + a}
\]

which is the centroid of the points \( A \) and \( B \) with associated numbers \( b \) and \( a \) respectively and therefore lies in \( AB \), dividing it into

\( b : a \) or ratio \( BC : AC \).

---

1 C. E. Weatherburn, *op. cit.*, pp. 9-10.
Corollary. If the end-points of the sides of the exterior angle of a triangle are joined together, the bisector of that angle divides the line joining the end-points into segments which are proportional to the sides of the angle, i. e.,

\[ b : a :: BN : AN \]

where CN is the bisector (figure 15).

Fig. 15.

Summary Chapter II. The vector equation of a circle

a. with reference point at center is

\[ \vec{r}^2 - \vec{a}^2 = 0. \]

b. with reference point outside of circle is

\[ \vec{r}^2 - 2\vec{r} \cdot \vec{c} = \vec{a}^2 - \vec{c}^2. \]

c. with reference point on circle is

\[ \vec{r}^2 - 2\vec{r} \cdot \vec{a} = 0. \]

The vector equation of the bisector of the angle between two straight lines OA and OB, parallel to unit vectors \( \vec{a} \) and \( \vec{b} \) respectively is

\[ \vec{r} = t (\vec{a} + \vec{b}) ; \]
and the bisector of its supplement is
\[ \overrightarrow{r} = t (\overrightarrow{a} - \overrightarrow{b}). \]
The position vector of \( P \) for the origin \( 0 \) is \( \overrightarrow{OF} \).

Given \( n \) points whose position vectors relative to an origin \( 0 \) are \( \overrightarrow{a}, \overrightarrow{b}, \ldots, \) the point \( P \) whose position vector is
\[ \overrightarrow{OF} = \frac{p \overrightarrow{a} + q \overrightarrow{b} + \cdots}{p + q + \cdots} \]
where \( p, q, \ldots \) are real numbers, is called the centroid or center of mean position.

The centroid of two points \( A, B \) with associated numbers \( p, q \), divides the line \( AB \) in the ratio \( q : p \) (\( q \) and \( p \) may be positive or negative).

The above equation and principles along with fundamental operations introduced in Chapter I have been combined and applied to familiar theorems in plane geometry.
CHAPTER III

VECTOR METHODS IN PLANE TRIGONOMETRY

The fundamental formulas of plane trigonometry as well as other important and familiar formulas and laws follow immediately from the scalar and vector products of vector analysis introduced in Chapter I.

The fundamental formulas in trigonometry are those for the sine and cosine of the difference of two angles and those for the sine and cosine of the sum of two angles, namely:

\[
\begin{align*}
\cos (\phi - \theta) &= \cos \phi \cos \theta + \sin \phi \sin \theta \\
\cos (\phi + \theta) &= \cos \phi \cos \theta - \sin \phi \sin \theta \\
\sin (\phi - \theta) &= \cos \phi \sin \theta - \cos \theta \sin \phi \\
\sin (\phi + \theta) &= \cos \phi \sin \theta + \cos \theta \sin \phi
\end{align*}
\]
Formula 1.

To prove: \( \cos(\phi - \theta) = \cos \phi \cos \theta + \sin \phi \sin \theta \).

Given: any vectors \( \overline{a} \) and \( \overline{b} \), in \( \overline{T} \overline{J} \) - plane; \( \overline{a} \) makes \( \theta \) with \( \overline{T} \) (figure 16), and \( \overline{b} \) makes \( \phi \) with \( \overline{T} \).

\[
\begin{align*}
\cos \theta &= \frac{\overline{M}}{|\overline{a}|} = \frac{x_1}{|\overline{a}|}, & x_1 = |\overline{a}| \cos \theta \quad \text{---------(19)} \\
\sin \theta &= \frac{\overline{N}}{|\overline{a}|} = \frac{y_1}{|\overline{a}|}, & y_1 = |\overline{a}| \sin \theta \quad \text{---------(20)} \\
\cos \phi &= \frac{\overline{P}}{|\overline{b}|} = \frac{x_2}{|\overline{b}|}, & x_2 = |\overline{b}| \cos \phi \quad \text{---------(21)} \\
\sin \phi &= \frac{\overline{Q}}{|\overline{b}|} = \frac{y_2}{|\overline{b}|}, & y_2 = |\overline{b}| \sin \phi \quad \text{---------(22)}
\end{align*}
\]

Also
\[
\overline{a} = \overline{M} + \overline{N}, \quad \overline{b} = \overline{P} + \overline{Q}.
\]

By (19) and (20),
\[
\overline{a} = |\overline{a}| \cos \theta \overline{T} + |\overline{a}| \sin \theta \overline{J}.
\]

By (21) and (22),
\[
\overline{b} = |\overline{b}| \cos \phi \overline{T} + |\overline{b}| \sin \phi \overline{J}.
\]

By (19),
\[
\overline{a} \cdot \overline{b} = |\overline{a}| |\overline{b}| \cos \theta \cos \phi + |\overline{a}| |\overline{b}| \sin \phi \sin \theta.
\]

By (7),
\[
\overline{a} \cdot \overline{b} = \overline{a} \cdot \overline{b} = ab \cos (\overline{a}, \overline{b}) = ab \cos (\phi - \theta).
\]

\[
\therefore \cos (\phi - \theta) = \cos \phi \cos \theta + \sin \phi \sin \theta.
\]
Formula 2.

To prove: \( \cos (\phi + \Theta) = \cos \phi \cos \Theta - \sin \phi \sin \Theta \).

Given: any two vectors, \( \overrightarrow{a} \) and \( \overrightarrow{b} \), in the \( \overrightarrow{I} \overrightarrow{J} \) - plane;
\( \overrightarrow{a} \) makes an angle \( -\Theta \) with \( \overrightarrow{I} \) and \( \overrightarrow{b} \) makes an angle \( +\phi \)
with \( \overrightarrow{I} \) (figure 17).

\[
\begin{align*}
\overrightarrow{a} &= x_1\overrightarrow{I} - y_1\overrightarrow{J} \quad - - - - - - - - - - (32) \\
\overrightarrow{b} &= x_2\overrightarrow{I} + y_2\overrightarrow{J} \quad - - - - - - - - - - - (33)
\end{align*}
\]

By (24), \( \sin (-\Theta) = \frac{y_1}{|a|} \), \( -y_1 = |a|\sin (-\Theta) \), \( y_1 = |a|\sin\Theta \) \( - - - - - - - - - - - (30) \)

By (25), \( \cos (-\Theta) = \frac{x_1}{|a|} \), \( x_1 = |a|\cos(-\Theta) = |a|\cos \Theta \) \( - - - - - - - - - - - (31) \)

\[
\overrightarrow{a} = x_1\overrightarrow{I} - y_1\overrightarrow{J} \quad - - - - - - - - - - - (32) \\
\overrightarrow{b} = x_2\overrightarrow{I} + y_2\overrightarrow{J} \quad - - - - - - - - - - - - (33)
\]

By (30), (31), \( \overrightarrow{a} = |a| \cos \Theta \overrightarrow{I} - |a|\sin \Theta \overrightarrow{J} \).

By (28), (29), \( \overrightarrow{b} = |b| \cos \phi \overrightarrow{I} + |b| \sin \phi \overrightarrow{J} \).

By (9), \( \overrightarrow{a} \cdot \overrightarrow{b} = ab \cos \Theta \cos \phi - ab \sin \Theta \sin \phi \).

By (7), (25), \( \overrightarrow{a} \cdot \overrightarrow{b} = ab \cos (\overrightarrow{a}, \overrightarrow{b}) = ab \cos (\phi + \Theta) \).

\[
\therefore \quad \cos (\phi + \Theta) = \cos \Theta \cos \phi - \sin \phi \sin \Theta .
\]
Formula 3.

To prove: \( \sin(\phi - \Theta) = \sin \phi \cos \Theta - \sin \Theta \cos \phi \).

Given: any vectors, \( \overrightarrow{a} \) and \( \overrightarrow{b} \), in the \( \mathbb{I} \mathbb{J} \)-plane;

\( \overrightarrow{a} \) makes \( \angle \Theta \) with \( \mathbb{I} \) and \( \overrightarrow{b} \) makes \( \angle \Phi \) with \( \mathbb{I} \) (figure 16).

By (23) and (24), \( \overrightarrow{a} \times \overrightarrow{b} = (x_1\mathbb{I} + y_1\mathbb{J}) \times (x_2\mathbb{I} + y_2\mathbb{J}) \)

By (13), \( = (x_1y_2k - y_1x_2k) \)

By (19) - (22), \( = (\overrightarrow{e} \cos \Theta \sin \phi - \overrightarrow{e} \sin \Theta \cos \phi) \)

By (10), \( \overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{e} \sin(\overrightarrow{a}, \overrightarrow{b}) = \overrightarrow{e} \sin(\phi - \Theta) \)

\( \therefore \sin(\phi - \Theta) = \cos \Theta \sin \phi - \cos \phi \sin \Theta. \)

Formula 4.

To prove: \( \sin(\phi + \Theta) = \sin \phi \cos \Theta + \cos \phi \sin \Theta. \)

Given: any vectors, \( \overrightarrow{a} \) and \( \overrightarrow{b} \), in the \( \mathbb{I} \mathbb{J} \)-plane;

\( \overrightarrow{a} \) makes an angle \( (-\Theta) \) with \( \mathbb{I} \), and \( \overrightarrow{b} \) makes an angle \( \Phi \) with \( \mathbb{I} \) (see figure 17).

By (32) and (33), \( \overrightarrow{a} \times \overrightarrow{b} = (x_1\mathbb{I} - y_1\mathbb{J}) \times (x_2\mathbb{I} + y_2\mathbb{J}) \)

By (13), \( = x_1y_2k + y_1x_2k \)

By (28) - (31), \( = \overrightarrow{e} \cos \Theta \sin \phi + \overrightarrow{e} \sin \Theta \cos \phi \)

By (10), \( \overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{e} \sin(\overrightarrow{a}, \overrightarrow{b}) \)

\( = \overrightarrow{e} \sin(\phi + \Theta) \)

\( \therefore \sin(\phi + \Theta) = \sin \phi \cos \Theta + \sin \Theta \cos \phi. \)

Formula 5.

To prove: \( \sin(\phi + \Theta) + \sin(\phi - \Theta) = 2 \sin \phi \cos \Theta. \)

The above formula follows from formulas (3) and (4) upon performing the indicated addition.
Formula 6.

To prove: \( \sin(\phi + \Theta) - \sin(\phi - \Theta) = 2 \cos \phi \sin \Theta \).

The above formula follows from formulas (3) and (4) upon performing the indicated subtraction.

Formula 7.

To prove: \( \cos(\phi + \Theta) + \cos(\phi - \Theta) = 2 \cos \phi \cos \Theta \).

The above formula follows from formulas (1) and (2) upon performing the indicated addition.

Formula 8.

To prove: \( \cos(\phi + \Theta) - \cos(\phi - \Theta) = -2 \sin \phi \sin \Theta \).

The above formula follows from formulas (1) and (2) upon performing the indicated subtraction.

Formula 9.

To prove: \( \tan(\phi + \Theta) = \frac{\tan \phi + \tan \Theta}{1 - \tan \phi \tan \Theta} \).

Since \( \tan(\phi + \Theta) = \frac{\sin(\phi + \Theta)}{\cos(\phi + \Theta)} \),
then, by formulas (4) and (2),

\[
\tan(\phi + \Theta) = \frac{\sin \phi \cos \Theta + \cos \phi \sin \Theta}{\cos \phi \cos \Theta - \sin \phi \sin \Theta}.
\]

Upon dividing each term of numerator and denominator by \( \cos \phi \cos \Theta \),

we obtain

\[
\tan(\phi + \Theta) = \frac{\tan \phi + \tan \Theta}{1 - \tan \phi \tan \Theta}.
\]
Formula 10.

To prove: \( \tan (\phi - \theta) = \frac{\tan \phi - \tan \theta}{1 + \tan \phi \tan \theta} \).

Since \( \tan (\phi - \theta) = \frac{\sin (\phi - \theta)}{\cos (\phi - \theta)} \), then by formulas (3) and (1),

\[
\tan (\phi - \theta) = \frac{\sin \phi \cos \theta - \cos \phi \sin \theta}{\cos \phi \cos \theta + \sin \phi \sin \theta}.
\]

Upon dividing each term of numerator and denominator by \( \cos \phi \cos \theta \), we get

\[
\tan (\phi - \theta) = \frac{\tan \phi - \tan \theta}{1 + \tan \phi \tan \theta}.
\]

Formula 11.

To prove: \( \cot (\phi + \theta) = \frac{\cot \phi \cot \theta - 1}{\cot \phi + \cot \theta} \).

Since \( \cot (\phi + \theta) = \frac{\cos (\phi + \theta)}{\sin (\phi + \theta)} \), then by formulas (2) and (4), we find

\[
\cot (\phi + \theta) = \frac{\cos \phi \cos \theta - \sin \phi \sin \theta}{\sin \phi \cos \theta + \sin \theta \cos \phi}.
\]

Upon dividing each term of numerator and denominator by \( \sin \phi \sin \theta \), we obtain

\[
\cot (\phi + \theta) = \frac{\cot \phi \cot \theta - 1}{\cot \phi + \cot \theta}.
\]

Formula 12.

To prove: \( \cot (\phi - \theta) = \frac{\cot \phi \cot \theta - 1}{\cot \phi - \cot \theta} \).

Since \( \cot (\phi - \theta) = \frac{\cos (\phi - \theta)}{\sin (\phi - \theta)} \), then by formulas (1) and (3), we find

\[
\cot (\phi - \theta) = \frac{\cos \phi \cos \theta + \sin \phi \sin \theta}{\sin \phi \cos \theta - \sin \theta \cos \phi}.
\]

by dividing each term of numerator and denominator by \( \sin \phi \sin \theta \), we finally get

\[
\cot (\phi - \theta) = \frac{\cot \phi \cot \theta + 1}{\cot \phi - \cot \theta}.
\]
Law 1.

To verify: \( c^2 = a^2 + b^2 - 2ab \cos(a, b) \); or, the square of one side of a triangle is equal to the sum of the squares of the other two sides diminished by twice the product of either of those sides by the projection of the other upon it—Law of Cosines—(figure 18).

\[ \overrightarrow{c} = \overrightarrow{a} - \overrightarrow{b} \]

\[ \overrightarrow{c} \cdot \overrightarrow{c} = (\overrightarrow{a} - \overrightarrow{b}) \cdot (\overrightarrow{a} - \overrightarrow{b}) \]

\[ = \overrightarrow{a} \cdot \overrightarrow{a} - 2\overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{b} \cdot \overrightarrow{b} \]

\[ = \overrightarrow{a} \cdot \overrightarrow{a} - 2ab \cos(\overrightarrow{a}, \overrightarrow{b}) + \overrightarrow{b} \cdot \overrightarrow{b} , \]

or

\[ c^2 = a^2 + b^2 - 2ab \cos(a, b) . \]

Law 2.

To verify: \( c^2 - d^2 = 4ab \cos(a, b) \),
or, the difference of the squares of the diagonals of a parallelogram is equal to four times the product of one of the sides by the projection of the other upon it.
then
\[ \overline{c} = \overline{a} + \overline{b} \]
\[ \overline{d} = \overline{a} - \overline{b}; \]

then
\[ \overline{c} \cdot \overline{c} = \overline{a} \cdot \overline{a} + 2\overline{a} \cdot \overline{b} + \overline{b} \cdot \overline{b} \]
\[ \overline{d} \cdot \overline{d} = \overline{a} \cdot \overline{a} - 2\overline{a} \cdot \overline{b} + \overline{b} \cdot \overline{b}; \]
\[ \overline{c} \cdot \overline{c} - \overline{d} \cdot \overline{d} = 4 \overline{a} \cdot \overline{b} = 4 \, ab \cos (\overline{a}, \overline{b}); \]
or,
\[ c^2 - d^2 = 4 \, a \cdot b \cos (a, b). \]

Summary of Chapter III. In chapter III formulas of vector analysis have been applied to verify formulas of Plane Trigonometry.

The following formulas and laws have been proved:

1. Sine and Cosine of Sum of Two Angles.
2. Sine and Cosine of Difference of Two Angles.
3. Sum of the Sines of the Sum and Difference of Two Angles.
4. Difference of the Sines of the Sum and Difference of Two Angles.
5. Sum of the Cosines of the Sum and Difference of Two Angles.
6. Difference of the Cosines of the Sum and Difference of Two Angles.
7. Tangent and Cotangent of Sum of Two Angles.
8. Tangent and Cotangent of Difference of Two Angles.
CHAPTER IV

CONCLUSIONS

Vector algebra and vector geometry furnish a powerful instrument in plane geometry and plane trigonometry although this is only a part of the domains in which it renders its greatest service. These domains of greatest service are those of mechanics and mathematical physics.

Scalar or dot products and vector or cross products along with general properties of mathematical vectors and some laws of scalar quantities have made this rendition possible.

It was not deemed necessary to make all possible applications of vector methods to the above named branches of mathematics, but enough to prove conclusively its utility in these subjects.
BIBLIOGRAPHY


