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Lectures in advanced calculus

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LECTURES IN ADVANCED CALCULUS

A THESIS

SUBMITTED TO THE FACULTY OF ATLANTA UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF MASTER OF SCIENCE

BY

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INTRODUCTION

In the field of mathematics, students usually do not encounter or begin to develop precise analytical proofs which contain several members of the Greek alphabet until the student has entered a course in Advanced Calculus. Since logical, accurate, analytical proofs are stressed here rather than mechanical manipulation, the unlimited usage of the Greek alphabet and the vigorous treatment of proofs tend to present a problem of adjustment to the student.

This thesis was written with the hope of clarifying some of those problems. Some of the more important theorems and concepts are presented with proofs that are sometimes abbreviated in several textbooks.

The thesis is divided into three main topics dealing with differentiation, series, and integration respectively. Generally, definitions and theorems are given without examples since any standard textbook will contain a sufficient amount of examples to illustrate the theorem or definition in question. The work presented herein is usually studied during the second semester of graduate work in advanced Calculus. The writer hopes that the reader will find some of the more difficult concepts and theorems encountered in mathematics somewhat simplified.
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CHAPTER I

DIFFERENTIATION

The Derivative.---A function $y = f(x)$ is said to have a derivative
or be differentiable at a point $x$ if and only if
\[
\frac{dy}{dx} \equiv f'(x) \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}
\]
exists and is finite. The functional value $f'(x)$ defined by the limit is called its
1 derivative.

The above definition may be stated in another way, that is, the derivative of a function is the limit of the ratio of the increment of the function to the increment of the independent variable when the independent variable varies and approaches zero as a limit. Symbolically this means:

(i). Given a functional value (*.*) $y = f(x)$ and let $x$
be fixed. Let $x$ take on the increment $\Delta x$.
Then $y$ takes on the increment $\Delta y$ giving the
new functional value (**.*) $y + \Delta y = f(x+\Delta x)$.

(ii). Subtracting (*.) and (**.*) we get $\Delta y = f(x+\Delta x) - f(x)$.

(iii.) Dividing by $\Delta x$ we then get
\[
\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}.
\]

Let $\Delta x \to 0$ and the limit (if it exists) of the right-hand member as $\Delta x \to 0$ denoted by $\frac{dy}{dx}$ is, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$, which defines the derivative of $f$ at the point $x$.

Note that

1. If $\Delta x$ and $\Delta y$ have like signs, then as the increment of the independent variable increases the increment of the dependent variable increases and vice versa.

2. If $\Delta x$ and $\Delta y$ have opposite signs, then as the increment of the independent variable increases, the increment of the dependent variable decreases or vice versa.

3. Differentiability implies continuity but continuity does not imply differentiability.¹

Theorem I. Chain Rule.

If $\gamma$ is a differentiable function of $u$ and if $u$ is a differentiable function of $x$, then $\gamma$, as a function of $x$, is differentiable and $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Proof: Let $\gamma = \check{g}(u)$ be differentiable at $u = u_0 = f(x_0)$ and let $u = f(x)$ be differentiable at $x = x_0$. As $x$ takes on the increment $\Delta x$, $u$ will take on the increment $\Delta u$. Hence

(1) $\gamma \Delta u = \check{g}(u + \Delta u)$ and (2) $u + \Delta u = f(x + \Delta x)$.

¹J. M. H. Olmsted, op. cit., p. 86.
Subtracting (\#) from (1) and dividing by \( \Delta u \) we get (3) \( \frac{\Delta y}{\Delta u} \)
\[
\frac{\Delta y}{\Delta u} = \frac{\varepsilon(u + \Delta u) - \varepsilon(u)}{\Delta u}. \quad \text{Subtracting (\#\#) from (2) and dividing by \( \Delta x \)}
\]
we get (4) \( \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \). By combining (3) and (4)

we now get \( \frac{\Delta y}{\Delta x}, \frac{\Delta u}{\Delta u} = \frac{\Delta y}{\Delta x} \frac{\Delta u}{\Delta u} = \frac{\Delta y}{\Delta x} \). Therefore

\[
\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}, \text{ and since is a continuous function}
\]
of \( x \) at \( x = a \), then \( \Delta x \rightarrow 0 \) implies \( \Delta u \rightarrow 0 \). Define \( \Delta u \) by

\[
\varepsilon(\Delta u) = \begin{cases} \frac{\Delta y}{\Delta u} - \frac{\Delta y}{\Delta u} \; \text{if} \; \Delta u \neq 0 \\ 0 \; \text{if} \; \Delta u = 0 \end{cases}
\]

Then \( \lim_{\Delta x \to 0} \varepsilon(\Delta u) = \lim_{\Delta u \to 0} \varepsilon(\Delta u) = \frac{\Delta y}{\Delta u} - \frac{\Delta y}{\Delta u} = 0 \).

If \( \Delta u \neq 0 \), then from (\#\#) we get \( \frac{\Delta y}{\Delta u} = \varepsilon(\Delta u) \),

i.e., \( \Delta y = \frac{\Delta y}{\Delta u} \cdot \Delta u + \varepsilon(\Delta u) \cdot \Delta u \).

Now \( \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{dy}{du} \cdot \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta y - \Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}.

Therefore \( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} + 0 \cdot \frac{du}{dx} \). \footnote{Ibid., p. 88.}

Theorem II.

If \( y = f(x) \) is strictly monotonic and differentiable

in an interval and if \( f'(x) \neq 0 \) in this interval, then the inverse

function \( x = \phi(y) \) is strictly monotonic and differentiable in

the corresponding interval and \( \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}. \)

Proof: We proceed along the same lines as the proof for

Theorem I. Let (\#)* \( y = f(x) \) and (\#\#)* \( x = \phi(y) \). Then by
letting (**) take on the increment \( \Delta y \) and (***) take on the increment \( \Delta x \) we get the new functional value (1) \( y + \Delta y = f(x + \Delta x) \),
and (2) \( x + \Delta x = \Phi(y + \Delta y) \) Subtracting (***) from (1) and (****) from (2) we now have (3) \( \Delta y = f(x + \Delta x) - f(x) \) and (4) \( \Delta x = \Phi(y + \Delta y) - \Phi(y) \).

If we divide (3) by the increment \( \Delta x \) and (4) by the increment \( \Delta y \)
we get (5) \( \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \) and (6) \( \frac{\Delta x}{\Delta y} = \frac{\Phi(y + \Delta y) - \Phi(y)}{\Delta y} \).

Combining (5) and (6) we have \( \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta y} = \frac{\Delta y \Delta x}{\Delta y \Delta x} = 1 \) or \( \frac{\Delta x}{\Delta y} = \frac{\Delta y}{\Delta x} \).

Since \( \Phi(y) \) is a continuous function, then \( \Delta x \to 0 \) if and only if \( \Delta y \to 0 \).

Therefore \( \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} = \frac{1}{\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta y}} \), Hence \( \frac{dy}{dx} = \frac{\Delta y}{\Delta x} \).

**Left-sided and Right-sided Derivatives.**

**Definition 1.** The derivative from the right of a function \( f \)
at a point \( x = a \) is the one-sided limit \( \lim_{h \to 0^+} \left\{ \frac{f(a + h) - f(a)}{h} \right\} \).

**Definition II.**

The derivative from the left of a function \( f \) at a point \( x = a \)
is the one-sided limit \( \lim_{h \to 0^-} \left\{ \frac{f(a + h) - f(a)}{h} \right\} \).

**Definition III.**

The right-hand derivative of a function \( f \) at the point \( x = a \)

1. Ibid.
2. Ibid., p. 89.
3. Ibid.
is the one-sided limit \( \lim_{h \to 0^+} \left\{ \frac{f(a + h) - f(a^+)}{h} \right\} \).  

Definition IV.

The left-hand derivative of a function at the point \( x = a \) is the one-sided limit \( \lim_{h \to 0^-} \left\{ \frac{f(a - h) - f(a^-)}{h} \right\} \).

Definition V.

The right-hand limit of the derivative of a function \( f \) at the point \( x = a \) is the one-sided limit \( f'(a^+) \equiv \lim_{x \to a^+} f'(x) \).

Definition VI.

The left-hand limit of the derivative of a function \( f \) at the point \( x = a \) is the one-sided limit \( f'(a^-) \equiv \lim_{x \to a^-} f'(x) \).

Note. A function has a derivative at a point if and only if it has equal derivatives from the right and from the left at the point.

Theorem III.

If \( f(x) \) is continuous on the closed interval \([a, b]\) and differentiable in the open interval \((a, b)\) and if \( f(x) \) assumes

---

1. Ibid.
2. Ibid.
3. Ibid.
4. Ibid.
either its maximum or minimum value for the closed interval \([a, b]\)
at an interior point \(\xi\) of the interval, then \(f'(\xi) = 0\).

Proof: Assume the hypothesis of the theorem and, for
definiteness, let \(f(\xi)\) be the maximum values of \(f(x)\) for the
interval \([a, b]\) where \(a < \xi < b\). Consider the difference quotient
\[
\frac{\Delta y}{\Delta x} = \frac{f(\xi + \Delta x) - f(\xi)}{\Delta x}
\]
for values of \(\Delta x\) so small numerically
that \(\xi + \Delta x\) is also in the open interval \((a, b)\). Since \(f(\xi)\)
is the maximum value of \(f(x)\), \(f(\xi) \geq f(\xi + \Delta x)\) and \(\Delta y \leq 0\).
Therefore \(\frac{\Delta y}{\Delta x} \leq 0\) for \(\Delta x < 0\) and \(\frac{\Delta y}{\Delta x} \geq 0\) for \(\Delta x > 0\).
Hence (in the limit), the derivative from the left at \(\xi\)
is non-negative and the derivative from the right at \(\xi\) is
non-positive. By hypothesis these one-sided derivatives are
equal and must therefore both equal zero.¹

Now, let \(f(\xi)\) be the minimum value of \(f(x)\) for the closed
interval \([a, b]\) where \(a < \xi < b\). Consider the difference quotient
\[
\frac{\Delta y}{\Delta x} = \frac{f(\xi + \Delta x) - f(\xi)}{\Delta x}
\]
for values of \(\Delta x\) so small numerically
that \(\xi + \Delta x\) is also in the open interval \((a, b)\). Since \(f(\xi)\)
is the minimum value of \(f(x)\), \(f(\xi) \leq f(\xi + \Delta x)\) and \(\Delta y \geq 0\). Therefore
\(\frac{\Delta y}{\Delta x} \leq 0\) for \(\Delta x < 0\) and \(\frac{\Delta y}{\Delta x} \geq 0\) for \(\Delta x > 0\). Hence, (in the limit),
the derivative from the left at \(\xi\) is non-positive and the
derivative from the right at \(\xi\) is non-negative. By hypothesis
these one-sided derivatives are equal and must therefore both
equal zero.

¹
Ibid., p. 93.
Theorem IV. Rolle's Theorem.

If a function \( f \) is continuous on the closed interval \([a, b]\) and if \( f'(x) \) exist in the open interval \((a, b)\) and if 
\[ f'(a) = f'(b) = 0, \]
then there exist a point \( \xi \in (a, b) \) such that
\[ f' \left( \xi \right) = 0. \]

Proof: If \( f(x) = 0 \) on \([a, b]\) then the proof is trivial. If \( f(x) \neq 0 \) then somewhere in the interval \([a, b]\) it is positive or negative and reaches a maximum or minimum there. Therefore Theorem III applies and the proof is complete.

Theorem V. The Mean Value Theorem #1.

If \( f(x) \) is continuous on the closed interval \([a, b]\) and \( f'(x) \) exists in the open interval \((a, b)\) then there exist a point \( \xi \in (a, b) \) such that 
\[ f(b) - f(a) = f'(\xi)(b - a) \]
or
\[ f'(\xi) = \frac{f(b) - f(a)}{b - a}, \]
assuming \( a \neq b \) and \( a < b \). This is actually a generalization of Rolle's Theorem.

Theorem VI. The General Mean Value Theorem.

If \( f(x) \) and \( g(x) \) are continuous on the closed interval \([a, b]\) and if \( f'(x) \) and \( g'(x) \) exists in the open interval \((a, b)\), then there exists a point \( \xi \in (a, b) \) such that
\[ g'(\xi) \left[ f(b) - f(a) \right] = f'(\xi) \left[ g(b) - g(a) \right]. \]

Proof: Consider the function \( F(x) \) defined by 
\[ F(x) = f(x) - K \cdot g(x), \]
where \( K \) is to be determined. \( F \) is continuous since \( f(x) \) and \( g(x) \) are continuous on the open interval \((a, b)\) and \( F(x) \) exists in the open interval \((a, b)\). We want \( K \) such that 
\[ f(b) - f(a) = K \left[ g(b) - g(a) \right]. \]
That is
\[ f(b) - f(a) = K \left[ g(b) - g(a) \right]. \]
Suppose that $g(b) \neq g(a)$. Then
\[ \kappa = \frac{f(b) - f(a)}{g(b) - g(a)}. \]
Thus in this case, the hypothesis of Rolle's Theorem, relative to $F$ are satisfied. Hence, there exists a point $\xi$ in the open interval $(a, b)$ such that $F'(\xi) = 0$. That is, $F'(\xi) = 0$.

Hence the result. If $g(a) = g(b)$ then we apply Rolle's Theorem to the function $g$. Hence there exists a point $\xi$ in the open interval $(a, b)$ such that $g'(\xi) = 0$, and the theorem again holds. Q.E.D.

Meaning of the Mean Value Theorem.

In figure 1, let slope $\ell = \frac{f(b) - f(a)}{b - a}$ and slope $\ell' = f'(\xi)$, $\ell'$ is tangent to $C$ and parallel to $\ell$ and $\ell'$ parallel to 1 implies $f'(\xi) = \frac{f(b) - f(a)}{b - a}$. Between the derivative $\frac{dy}{dx} = f'(x)$ and the difference quotient there exists a simple relation which is important for many purposes. This relation is known as the Mean Value Theorem and is obtained in the following way. We consider
the difference quotient \( \frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a} \) of a function \( f(x) \) and assume that the derivative exists everywhere in the interval \( a \leq x \leq b \), so that the graph of the curve has a tangent everywhere. The difference quotient will be represented by the direction of the secant. Let us imagine this secant shifted parallel to itself. At least once, it will reach a position in which it is tangent to the curve at a point between \( a \) and \( b \), namely at the point of the curve which is at the greatest distance from the secant. Hence there will be an intermediate value \( F \) such that \( f(\xi) = \frac{f(b) - f(a)}{b - a} \). This is known as the mean value theorem of Integral Calculus.

Geometric Interpretation of the General Mean Value Theorem.

In figure 2, consider the curve \( C \) in parametric form:

1 R. Courant, Differential and Integral Calculus, New York, 1942, p. 72.
where \( a \leq t \leq b \). Let slope \( l = \frac{f(b) - f(a)}{b - a} \)
and slope \( l' = \frac{dy}{dx} \bigg|_{t=x} = \frac{f'(x)}{g'(x)} \). \( l' \) parallel to \( l \) implies

\[
\frac{f'(x)}{g'(x)} = \frac{f(b) - f(a)}{g(b) - g(a)}
\]

Let \( f \) be continuous in a neighborhood \( U \) of a point \( x = a \) and let \( f'(x) \) exist continuous in that neighborhood. Then there exists a point 
\( \xi \in (a, b) \), where \( b \in U \) and \( a < b \) such that \( f'(\xi) = \frac{f(b) - f(a)}{b - a} \),
or \( f(b) = f(a) + f'(\xi)(b - a) \). Set \( h = b - a \). Then \( b = h + a \) and
\( \xi = a + \theta h \), where \( 0 < \theta < 1 \). Hence \( f(b) = f(a) + f'(\xi)(b - a) \)
becomes \( f(a + h) = f(a) + h(f'(a) + \theta h) \) where \( 0 < \theta < 1 \).

Consequences of the Mean Value Theorem.

**Theorem VI.**

If a function has zero derivatives throughout an interval, then the function is constant in that interval.

Proof: Let the function be denoted by \( f \). Let \( f'(x) = 0 \)
throughout the closed interval \([a, b]\). Suppose there exist two points
\( c \) and \( d \) belonging to the closed interval \([a, b]\) and \( c < d \) such that
\( f(c) \neq f(d) \). Then by the Mean Value Theorem there exists a
point \( \xi \) such that \( f'(\xi) = \frac{f(d) - f(c)}{d - c} \neq 0 \). But this contradicts the fact that \( f'(x) = 0 \) on \([a, b]\). Hence \( f(c) = f(d) \). Q.E.D.

**Theorem VII.**

If \( f(x) \) and \( g(x) \) have the same derivatives in the open interval
\((a, b)\), then \( f(x) \) and \( g(x) \) differ at most by an additive constant in \((a, b)\).
Proof: Let \( f'(x) = g(x) \) in \((a, b)\). Set \( h(x) = f(x) - g(x) \). Then \( h'(x) = f'(x) - g'(x) = 0 \) in \((a, b)\). Hence by the previous theorem, \( h(x) = c \) (where \( c \) is a constant).

Theorem VIII.

If \( f'(x) \) exist in \((a, b)\) and if \( f'(x) \geq 0 \) (\( \leq 0 \)), then \( f(x) \) is monotonically increasing (decreasing) in \((a, b)\). If \( f'(x) > 0 \) (\( < 0 \)), then \( f(x) \) is strictly monotonically increasing (decreasing) in \((a, b)\).

Proof: Let \( x_1 \) and \( x_2 \) be any two points in \((a, b)\) with \( x_1 < x_2 \). By the mean value Theorem there exists a point \( \xi \) belonging to \((x_1, x_2)\) such that \( f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \).

Hence \( f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \), and the conclusion can be read from this expression.

Theorem IX.

If \( f \) has non-zero derivatives throughout an interval, then \( f(x) \) has the same sign throughout that interval, and hence is strictly monotonic there.

Proof: By virtue of Theorem III, it is sufficient to show that \( f'(x) \) has the same sign throughout the interval. Suppose \( f'(x) \neq 0 \) in \((a, b)\) where \( a < b \). Suppose there exist points \((c, d)\), where \( c < d \), which belong to the interval \((a, b)\) such that \( f(c) \) and \( f(d) \) have different signs. By a previous theorem \( f(x) \) has a maximum and minimum on \([c, d]\). Let \( \xi \) be a point where \( f(x) \) takes its maximum and \( \eta \) a point where \( f(x) \) takes its minimum.
and \( \xi \neq \eta \), since \( f(x) \neq 0 \). \( \xi \) and \( \eta \) do not belong to the interval \((c, d)\), for by a previous theorem \( f'(\xi) = f'(\eta) = 0 \) which cannot be. Hence, either \( c = \xi \) and \( c = \xi \) or \( c = \eta \) and \( d = \xi \).

Case 1. \( c \) is a maximum.
Then \( f'(c) \leq 0 \) and \( f'(d) \leq 0 \).

Case 2. \( c \) is a minimum
Then \( f'(c) \geq 0 \) and \( f'(c) = 0 \).

In both cases we have a contradiction. Q.E.D.

Taylor's Theorem and The Extended Law of the Mean.

Definition. A function \( f \) is said to belong to class \( C^n \) on \((a, b)\) if and only if \( f^n(x) \) exists and is continuous in \((a, b)\), and \( n \geq 0 \).

Theorem X.
If \( f \) is defined on a closed interval \([a, b]\) and if \( f^{(n+1)}(x) \) exists in \((a, b)\) and if \( f^{(k)}(x) \) is continuous on \([a, b]\), and exists on \((a, b)\), then \( f(b) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!} (b-a)^k + R \), where \( R = \frac{f^{(n+1)}(\xi)}{(n+1)!} (b-a)^{n+1} \) and \( \xi \in (a, b) \).

If \( n = 0 \), then \( f(b) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!} (b-a)^k + R \) reduces to the Mean Value Theorem, for then we have, \( f(b) = f(a) + \frac{f^{(p+1)}(\xi)}{(p+1)!} (b-a)^{p+1} \).
So \( f(b) = f(a) + f'(\xi)(b-a) \) and this is a generalization of the Mean Value Theorem.

Proof: Let \( M \) be defined by
\[ f(b) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!} (b-a)^k + \frac{M(b-a)^{n+1}}{(n+1)!} \]
Set \( f(t) = -f(b) + f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!} (b-t)^k + M(b-t)^{n+1} \), where 
\[ a \leq t \leq b. \]
We wish to show that \( M = f^{(n+1)}(c) \) for \( c \in (a, b) \).

Observe that \( f(a) = f(b) = 0 \). Moreover \( F'(t) \) exists in \( (a, b) \).

Applying Rolle's Theorem, we see that there exists a point 
\( c \in (a, b) \) such that \( f'(c) = 0 \).

But \( F'(t) = f'(t) + \sum_{k=1}^{n} \left[ \frac{f^{(k+1)}(c)}{k!} (b-t)^k - \frac{f^{(k)}(c)}{(k-1)!} (b-t)^{k-1} \right] \) 
\[ \frac{M}{n!} (b-t)^n = \frac{f^{(n+1)}(c)}{n!} (b-t)^n - \frac{M}{n!} (b-t)^n. \]

Thus \( \frac{(b-t)^n}{n!} (f^{(n+1)}(c) - M) = F'(c) \), where \( c \in (a, b) \).

Thus \( f'(c) = 0 \) implies \( f^{(n+1)}(c) - M = 0 \) and this means 
\( f^{(n+1)}(c) = M \). Q.E.D.

L'Hospital's Rule and Indeterminate Forms.

Any expression of the form, \( (0 \cdot \infty), \frac{\infty}{\infty}, \frac{0}{0}, \frac{\infty}{0}, \infty \cdot \infty \), etc., is called an indeterminate form.

Theorem XI, L'Hospital's.

If \( f \) and \( g \) are differentiable in \( [a, b] \), if \( g'(x) \neq 0 \) in 
\( (a, b) \) and if 

(i).
\[ \lim_{x \to b^-} f(x) = 0 \quad \text{and} \quad \lim_{x \to b^-} g(x) = 0, \]

(ii).
\[ \lim_{x \to b^-} f(x) = \infty \quad \text{and} \quad \lim_{x \to b^-} g(x) = \infty, \]

(iii).
\[ \lim_{x \to b^-} \frac{f'(x)}{g'(x)} = L, \]
then \( \lim_{x \to b^-} \frac{f(x)}{g(x)} = L \).

Suppose \( g'(x) \neq 0 \) in \((a, b)\) and \( g \) is continuous on \([a, b]\).

Then we show that \( g'(a) \neq g'(b) \).

**Proof:** By the Mean Value Theorem, there exists a point \( \xi \in (a, b) \) such that
\[
\frac{g(b) - g(a)}{b - a} = g'({\xi}) = \frac{f(b) - f(a)}{g(b) - g(a)}.
\]
That is, \( g(b) - g(a) = g'({\xi})(b - a) \). But \((b - a) \neq 0\), (since \(a < b\)), and \( g'({\xi}) \neq 0 \). Hence \( g(b) - g(a) \neq 0 \) so \( g(b) \neq g(a) \).

Theorem XII.


f and \( g \) belong to \( c' \) on \([a, b]\), \( g'(x) \neq 0 \) in \((a, b)\) implies that there exists a point \( \xi \in (a, b) \) such that
\[
\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'({\xi})}{g'({\xi})}.
\]

**Proof of L'Hôpital's Rule:**

1. Under hypothesis (i). We assume \( b \) to be finite and

   Since \( \lim_{x \to b^-} f(x) \neq 0 \) and \( \lim_{x \to b^-} g(x) \neq 0 \), we can set \( f(b) = 0 \) and \( g(b) = 0 \). Then \( f \) and \( g \) are continuous on \([a, b]\).

   Applying the General Mean Value Theorem we see that there exists an \( x \) in \((x, b)\), where \( x \in (a, b) \) and \( a < x < b \)

   such that
   \[
   \frac{f'(x)}{g'(x)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f(x)}{g(x)} = \frac{f'({\xi})}{g'({\xi})}.
   \]

   Since \( \lim_{x \to b^-} \frac{f'(x)}{g'(x)} = L \), there exists a \( \delta \) such that

   \[
   \left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'({\xi})}{g'({\xi})} - L \right| < \frac{1}{n}, \quad \text{where} \quad b - \delta < x < b.
   \]

   Hence \( \lim_{x \to b^-} \frac{f(x)}{g(x)} = L \).
2. Under Hypothesis (ii). First given any $\varepsilon > 0$, choose $\delta$ such that \[ \left| \frac{f''(x)}{g'(x)} - L \right| < \varepsilon, \quad \text{for} \quad b - \delta < x < b. \]

Set $x_0 = b - \delta$. Then by the General Mean Value Theorem there exists a point $\bar{x}$ in $(x_0, x)$ such that

\[ \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\bar{x})}{g'(\bar{x})}. \]

Set $h(x) = \left\{ \begin{array}{ll} \frac{1 - f(x)}{g(x)} & \text{if } x > x_0, \\ \frac{1 - g(x)}{f(x)} & \text{if } x < x_0. \end{array} \right.$

Then observe that

\[ \left| f'(\bar{x}) - L \right| = \left| \frac{f(x) - f(x_0)}{g(x) - g(x_0)} - L \right| = \left| \frac{f(x) - f(x_0)}{g(x)} - \frac{g(x) - g(x_0)}{f(x)} \right| < \varepsilon \quad \text{for } b - \delta < \bar{x} < x < b. \]

Not that $h(x) + 1$ as $x \to b$ by virtue of hypothesis (ii). Hence there exists an $x_1$ in $(x_0, b)$ such that $h(x_1) > \frac{1}{2}$ and $|h(x) - 1| < \varepsilon$ for all $x_0 < x_1 < x < b$.

Then

\[ \left| \left( \frac{f(x)}{g(x)} - L \right) h(x) \right| = \left| \frac{f(x)}{g(x)} h(x) - h(x) \right| L \leq \left| \frac{f(x)}{g(x)} \right| h(x) L \]

whenever $x_0 < x < x < b$. Hence

\[ \left| \frac{f(x)}{g(x)} - L \right| h(x) \]

\[ < 1 + |L| \varepsilon < L \quad \text{wherever} \quad x_0 < x_1 < x < b. \]

Hence

\[ \lim_{x \to b^-} \frac{f(x)}{g(x)} = L. \]

Note. If $b = +\infty$ then we may define the functions $F$ and $G$ as follows: $F(x) = f(x)$ and $G(x) = g(x)$. $\frac{1}{x} \to 0^+$ as $x \to 0^+$ hence

\[ \lim_{t \to 0^+} \frac{f(t)}{g(t)} = \lim_{x \to 0^+} \frac{f(\frac{1}{x})}{g(\frac{1}{x})} = \lim_{x \to 0^+} \frac{F(x)}{G(x)}. \]

This case is reduced to the case just discussed (i) where $b$ is finite.
CHAPTER II

SERIES

Sequence.
Definition.

If \( a_1, a_2, \ldots, a_n, \ldots \) is a succession of terms formed by some fixed law then we call it a sequence. Briefly, a sequence is a function defined on the positive integers. We denote the set \( \{a_1, a_2, \ldots\} \) by \( \{a_n\} \) and call it a sequence, where \( \{a_n\} = a_1, a_2, \ldots, a_m, \ldots \). Each element of the sequence is called a term of the sequence, with \( a_n \) being called the general term.

Series.
Definition.

A series is the sum of the terms of a sequence. That is, if \( \{a_n\} \) is a series then \( (a_1 + a_2 + a_3 + \ldots) \) is a series which is denoted by \( \sum_{n=1}^{\infty} a_n \). If there is no possibility of confusion we write \( \sum a_n \) for an infinite series or, more briefly, a series.

Partial Sum.
Definition.

The sum of the first \( n \) terms of a series denoted by,
\[
S_n = a_1 + a_2 + a_3 + \ldots + a_n
\]
is a partial sum.

Convergent Series.
Definition.

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Every finite series converges and every infinite series converges if and only if \( \lim_{n \to \infty} S_n \) exists and is finite; otherwise the series diverges.

Examples.

1. Consider the series \( \sum_{n=1}^{\infty} n = 1 + 2 + 3 + \ldots \).
   
   \[ S_n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \quad \text{and} \quad \lim_{n \to \infty} S_n = +\infty. \]
   
   In this case, although the \( \lim_{n \to \infty} S_n \) exists, the series diverges to \(+\infty\).

2. Consider the series \( \sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + \ldots \).
   
   \[ S_n = \frac{1}{2} [1 + (-1)^{n-1}] \quad \lim_{n \to \infty} S_n \text{ does not exist in this case and hence the series does not converge.} \]

Theorem I.

The alternation of a finite number of the terms of a series does not affect convergence. The sum of the series, however, is affected.

Proof: Consider the series \( \sum_{n=1}^{\infty} a_n \) and assume convergence.

Suppose \( m \) terms of the series are altered. Since \( m \) is finite there exists a \( \mu \geq m+1 \) such that the terms \( a_{\mu} \) for \( \mu \leq n \) are not changed. Thus \( \sum_{n=1}^{\infty} a_n \to \sum_{n=1}^{\mu-1} a_n + \sum_{n=\mu}^{\infty} a_n + \sum_{n=\mu}^{\infty} b_n \)

is finite and hence it converges. Let \( S_n = a_1 + a_2 + a_3 + \ldots + a_n \),

and \( \overline{S}_n = a_\mu + a_{\mu+1} + a_{\mu+2} + \ldots + a_n \). \( \overline{S}_n = S_n - S_{\mu-1} \) and

\[ \lim_{n \to \infty} (S_n - S_{\mu-1}) = \lim_{n \to \infty} S_n - S_{\mu-1}. \] Hence \( \lim_{n \to \infty} \overline{S}_n \) exists and is finite, \( Q.E.D. \)

Monotone Sequences.

Definition.

A sequence \( \{a_n\} \) is said to be monotone increasing if for
each \( n \) we have an \( a_n \leq a_{n+1} \), where \( n = 1, 2, 3, \ldots \). The sequence is said to be monotone decreasing if \( a_n \geq a_{n+1} \), where \( n = 1, 2, 3, \ldots \).

If the sequence is monotone increasing or monotone decreasing it is called monotonic.

Definition.

A sequence is bounded if and only if all of its terms are contained in some interval. Equivalently, the sequence \( \{a_n\} \) is bounded if and only if there exists a positive number \( p \) such that \( |a_n| \leq p \) for all \( n \).

Definition.

Let \( \{a_n\} \) be a sequence. Then the number \( L \) is called the limit of the sequence, written \( \lim_{n \to \infty} a_n = L \), if and only if for any \( \varepsilon > 0 \), there exists an \( N \) such that \( |a_n - L| < \varepsilon \) for all \( n > N \).

Theorem II. The Cauchy Criterion for convergence of a sequence.

A necessary and sufficient condition for convergence of the sequence \( \{a_n\} \) is that, for any \( \varepsilon > 0 \) there exists an \( N = N(\varepsilon) \) such that \( |a_n - a_m| < \varepsilon \) for all \( n > N \) and \( m > N \).

Proof: Assume that the sequence \( \{a_n\} \) converges. Let \( L \) be the limit of the sequence \( \{a_n\} \). Then there exists for any \( \varepsilon > 0 \) an \( N = N(\varepsilon) \) such that \( |a_n - L| < \frac{\varepsilon}{2} \) and \( |a_m - L| < \frac{\varepsilon}{2} \), for all \( n, m > N \).

Observe that \( |a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |L - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \) for all \( n, m > N \). Hence the condition is necessary.

Assume the condition that \( |a_n - a_m| < \varepsilon \) for all \( n, m > N \).

Given \( \varepsilon > 0 \) there exists an \( N = N(\varepsilon) \) such that \( |a_n - a_m| \) for all \( n, m, > N \). Set \( m = N + 1 \), then \( |a_n - a_{n+1}| < \varepsilon \), for all \( n > N \).
Hence the sequence \( \{q_n\} \) is bounded. Therefore, by Theorem II.b, the sequence \( \{q_n\} \) has a limit point. Call this limit \( L \). Suppose there exists another limit point \( L' \) such that \(|L - L'| \neq 0\). Then set \(|L - L'| = 3\varepsilon \).

We know that there exists an \( N = N(\varepsilon) \) such that \(|a_n - L| < \frac{\varepsilon}{2} \) and \(|a_m - L'| < \frac{\varepsilon}{2}\) for all \( n, m > N(\varepsilon) \). Then observe that \( 3\varepsilon = |L - L'| = |L - a_n + a_n - a_m + a_m - L'| \leq |a_n - L'| + |a_m - a_n| + |a_n - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \varepsilon = 2\varepsilon \) for all \( n, m > N \). Thus we have that \( 3\varepsilon < 2\varepsilon \) which is a contradiction. Hence the condition is sufficient. Q.E.D.

Theorem II.a.

A necessary condition for convergence of a series is that the limit of the \( n^{th} \) term tends to zero.

Proof: Assume that the series \( \sum a_n \) converges. Then

\[
\lim_{n \to \infty} a_n = 0.
\]

Consider the partial sums \( S_n, S_m \). Observe that

\[
S_m - S_{n-1} = a_n.
\]

Suppose that \( L \) is the limit of the partial sums.

Then

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} (S_n - S_{n-1}) = \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1} = L - L = 0.
\]

Therefore \( \lim_{n \to \infty} a_n = 0 \). Q.E.D.

The above condition however is not sufficient. For example, consider the Harmonic Series \( \sum \frac{1}{n} \). Note that \( \lim_{n \to \infty} \frac{1}{n} = 0 \).

(i) The series \( \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \)

Hence the series \( \sum_{n=1}^{\infty} \frac{1}{n} (i) \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \)

The series \( 1 + \frac{3}{2} + \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \cdots \) which diverges to \( +\infty \).

Therefore, since the series \( \sum \frac{1}{n} \) is greater than the series represented in (ii), then the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.
Theorem II b.

Every monotone increasing sequence, bounded above has a limit. If a monotone increasing sequence is not bounded above, then it approaches $+\infty$ as a limit.

Proof: $\{q_\eta\}$ is not bounded above. Since $\{q_\eta\}$ is not bounded above, there exists for any positive number $M$, however large, a number $\eta_1$ such that $q_{\eta_1} > M$. But since $\{q_\eta\}$ is non-decreasing $q_n > M$ for all $\eta > \eta_1$, i.e. $\lim_{\eta \to \infty} q_n = +\infty$.

$\{q_\eta\}$ is Bounded above. Since $\{q_\eta\}$ is bounded above it has a least upper bound say $B$.

Then given $\epsilon > 0$ and arbitrary there exists an $\eta_0$ such that $B - \epsilon < q_{\eta_0} \leq B$, i.e. $B - \epsilon < q_\eta \leq B + \epsilon$. Since $\{q_\eta\}$ is non-decreasing we have $B - \epsilon < q_n \leq B + \epsilon$ for all $\eta > \eta_0$. i.e. adding $\epsilon$ to the inequality we get $-\epsilon < q_n - B \leq \epsilon$ for all $\eta > \eta_0$. Or $|q_n - B| < \epsilon$ for all $\eta > \eta_0$.

Theorem II c.

Any bounded monotonic sequence converges. If the sequence $\{q_\eta\}$ is monotone increasing and if $q_n \leq B$ for all $\eta$ then $\{q_\eta\}$ converges; moreover, if $q_n \geq B$ then $q_n \leq B + \epsilon$ for all $\eta$.

Proof: Since the sequence $\{q_\eta\}$ is bounded above then, by Theorem IIb, it has a least upper bound $B$ and for $\epsilon > 0$ there exists $\eta_0$ such that $|q_n - B| < \epsilon$ for all $\eta > \eta_0$. We wish to show that $q_n \to B$.

Observe that given $\epsilon > 0$ there exists a position integer $N$ such that $q_n \geq B - \epsilon$. Therefore $B - \epsilon < q_n \leq B < B + \epsilon$, i.e. $B - \epsilon < q_n < B + \epsilon$ or $|q_n - B| < \epsilon$ for all $\eta > N$.

Theorem III. Bolzano - Weierstrass Theorem.
Every infinite bounded set has at least one limit point.

Proof: Let $S$ be an infinite bounded set. Since it is bounded, there exists a closed interval $I$ such that $S \subset I$. Divide $I$ into two equal parts. At least one of these parts contains an infinite number of elements of $S$. Let this part be denoted by $I_1$. Divide $I_1$ into two equal parts. In one of the two parts there must be an infinite number of elements of $S$. Denote this part, $I_2$.

Continuing in this way, we obtain a sequence of non-increasing closed intervals $I_1, I_2, I_3, I_4, \ldots$. Each of the intervals contains infinitely many elements of $S$ and the length of the interval tends to zero as $n \to \infty$. Therefore $\bigcap_{n=1}^{\infty} I_n$ is a single point. Denote this point by $\eta$. Now, consider a neighborhood of $\eta$, i.e., $(\eta - \varepsilon, \eta + \varepsilon)$. There exists an interval $I_n$ such that $I_n \subset (\eta - \varepsilon, \eta + \varepsilon)$. The interval $I_n$ contains infinitely many elements of $S$ and therefore $(\eta - \varepsilon, \eta + \varepsilon)$ contains infinitely many elements of $S$. Hence $\eta$ is a limit point of $S$. Q.E.D.

Examples.

1. Consider the series $\sum_{n=1}^{\infty} \frac{n}{n(n+1)} = 1 + 2 + 3 + \ldots$.

   $S_n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$, and $\lim_{n \to \infty} S_n = +\infty$.

   In this case, although the $\lim_{n \to \infty} S_n$ exists, the series diverges to $+\infty$.

2. Consider the series $\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + 1 \ldots$.

   $S_n = \frac{1}{2} \left[ (1 + (-1)^{n-1}) \right]$. $\lim_{n \to \infty} S_n$ does not exist in this case and hence the series does not converge.

Theorem IV.
If a series converges then its sum is unique.

Proof: Suppose \( S_n \rightarrow L \) and \( S_n \rightarrow L' \) where \( L \neq L' \). Let \( \varepsilon = |L - L'| \).

Then \( \varepsilon = |L - S_n + S_n - L'| \leq |L - S_n| + |S_n - L'| \).

By assumption there exists \( N_1 = N(\varepsilon) \) such that \( |L - S_n| < \frac{\varepsilon}{2} \) and
\(|S_n - L'| < \frac{\varepsilon}{2} \) for all \( n > N \). Therefore \( |S_n - L_n + |S_n - L'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \), \( \varepsilon < 0 \) \( n \rightarrow N \).

This is a contradiction. Q.E.D.

Theorem V.

If all terms of a sequence, from some point on, are equal to a constant, the sequence converges to this constant.

Proof: Any neighborhood of the constant contains the constant and therefore all but a finite number of the terms of the sequence.

Theorem VI.

The sum of two convergent sequences is a convergent sequence, and the limit of the sum is the sum of the limits:

\[
\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n .
\]

This rule extends to the sum of any finite number of sequences.

Proof: Assume \( a_n \rightarrow a \) and \( b_n \rightarrow b \), let \( \varepsilon > 0 \) be given. Choose \( N \) so large that the following two inequalities hold simultaneously for \( n > N \): \( |b_n - b| < \frac{\varepsilon}{2} \) and \( |b_n - b| < \frac{\varepsilon}{2} \). Then for \( n > N \) \( \varepsilon > N \) we have
\[
\left| (a_n + b_n) - (a + b) \right| = \left| (a_n - a) + (b_n - b) \right| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .
\]

\[1\] Ibid., p. 34.
Theorem VII.

The product of two convergent sequences is a convergent sequence and the limit of the product is the product of the limits:

\[ \lim_{n \to \infty} (a_n b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n. \]

This rule extends to the product of any finite number of sequences.

Proof: Assume \( a_n \to a \) and \( b_n \to b \). We wish to show that \( a_n b_n \to ab \) or equivalently that \( a_n b_n - ab \to 0 \). By addition and subtraction of the quantity \( ab \) and by appeal to Theorem VI, we can use the relation

\[ a_n b_n - ab = (a_n - a)b_n + a(b_n - b) \]

to reduce the problem to that of showing that both sequences \( \{ (a_n - a)b_n \} \) and \( \{ a(b_n - b) \} \) converge to zero. The fact that they do is a consequence of the following lemma.

**Lemma:** If \( \{ a_n \} \) converges to zero and \( \{ b_n \} \) converges, then \( \{ a_n b_n \} \) converges to zero.

Proof of lemma: Since any convergent sequence is bounded, the sequence \( \{a_n\} \) is bounded and there exists a positive number \( \rho \) such that \( |a_n| \leq \rho \) for all \( n \). Given \( \epsilon > 0 \), choose \( N \) so large that

\[ |a_n| < \frac{\epsilon}{\rho} \text{ for all } n > N. \]

Then for \( n > N \),

\[ |a_n b_n| = |a_n| \cdot |b_n| < \frac{\epsilon}{\rho} \cdot |b_n| = \epsilon. \]

This inequality completes the proof of the lemma and hence the theorem.

Proof that any convergent sequence is bounded: Assume \( a_n \to a \) and choose a definite neighborhood of \( a \), say the open interval \((a-\epsilon, a+\epsilon)\). Since this neighborhood contains all but a finite number of terms of \( \{a_n\} \) a suitable enlargement will contain these
Theorem VIII.

The multiplication of the terms of a series by a constant does not affect convergence or divergence.

Proof: This is a consequence of Theorems V and VII.

Theorem IX. The Geometric Series.

A series of the form \( a + ar + ar^2 + \ldots \)

\[ \sum_{n=0}^{\infty} ar^n \] where \( a \neq 0 \) is called a Geometric Series. The Geometric series converges if \( |r| < 1 \) and diverges if \( |r| \geq 1 \).

Proof: Recall that

\[ S_n = \frac{a(1-r^n)}{1-r} \]

If \( |r| < 1 \), then we have two cases. Consider the case where \( |r| = 1 \).

This case is clear so we consider the case where \( |r| > 1 \). If \( |r| > 1 \), then \( S_n = \frac{a}{1-r}(1-\lim_{n \to \infty} r^n) = \pm \infty \).

Hence the Geometric Series diverges if \( |r| \geq 1 \).

Series of Positive Terms.

Consider series of positive terms or more generally series of non-negative terms, then the following theorems hold true.

Theorem X.

A series of non-negative terms is convergent if and only if the partial sums are bounded.

\[ \text{[Ibid.], p. 35.} \]
Proof: (i) Assume the series $\sum a_n = A$. Then for all $n$, $S_n \leq A$.

Therefore the partial sums are bounded.

(ii) Note that our requirement that $a_n > 0$ implies that $S_{n-1} \leq S_{n-1} + a_n = S_n$. Thus we have a bounded monotone increasing sequence of positive numbers and by Theorem IIb, every such series converges. Therefore the series is convergent. Q.E.D.

Theorem XI.

If $a_1 > a_2 > a_3 > \ldots a_j$ then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=0}^{\infty} a^2 k$ converges.

Proof: Set $S_n = a_1 + (a_1 + a_2) + (a_1 + a_2 + a_3) + \ldots + (a_1 + a_2 + \ldots + a_j)$.

(i) Consider the case where $n < 2^k$.

Then $S_n \leq a_1 + (a_1 + a_2) + \ldots + (a_1 + a_2 + \ldots + a_j)$.

(ii) Assume that $n > 2^k$.

Then $S_n \geq a_1 + a_2 + (a_1 + a_2 + a_3 + \ldots a_j) + \ldots + (a_1 + a_2 + \ldots + a_j)$.

which implies $S_n \geq \frac{1}{2} a_1 + a_2 + a_3 + \ldots + 2^{k-1} a_k = \frac{a_k}{2^k}$. Hence $S_n \geq \frac{a_k}{2^k}$. The theorem follows now by applying Theorem X and considering the above inequalities.

Theorem XII.

The series $\sum a_n$ converges if and only if there exists an $N = N(\varepsilon)$ such that $\left| \sum_{k=n}^{\infty} a_k \right| < \varepsilon$, for $n > N$.

Proof: Let $S_n = a_1 + a_2 + \ldots + a_n$. And $S_m = a_1 + a_2 + \ldots + a_m$.

Then by the Cauchy Criterion, given $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that $|S_m - S_n| < \varepsilon$, for $n, m > N$. Observe that
Theorem XIII.

The series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges if \( p > 1 \) and diverges if \( p \leq 1 \).

Proof: If \( p \geq 0 \) then the series diverges since \( \lim_{n \to \infty} \frac{1}{n^p} \neq 0 \).

If \( p > 0 \) we apply Theorem XI. Thus the series \( \sum_{k=0}^{\infty} 2^k q_{2k} \) converges if \( 2^{(1-p)} < 1 \) and if and only if \( 1 - p > 0 \) implies \( p > 1 \). Thus the series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges, by Theorem XI, if \( p > 1 \). But the series \( \sum_{k=0}^{\infty} 2^{(1-p)} \) diverges if \( (1-p) > 0 \), i.e., \( p \leq 1 \). Hence, by Theorem XI, the series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) diverges if \( 0 < p \leq 1 \). Q.E.D.

Theorem XIV.

The series \( \sum_{n=2}^{\infty} \frac{1}{n (\ln n)^p} \) converges if \( p > 1 \) and diverges if \( p \geq 1 \).

Proof: Consider the case where \( p > 0 \). If \( p > 0 \) Theorem XI applies and hence \( \sum_{k=0}^{\infty} 2^k a_{2k} \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k (\ln 2)^p \cdot k^p} \) which converges. If \( p > 0 \) then by applying Theorem II to the series \( \sum_{k=1}^{\infty} \frac{1}{(\ln 2)^p \cdot k^p} \) we see that the series diverges.
Definition.

The statement that a series \( \sum b_n \) dominates a series \( \sum a_n \) means that \( |a_n| \leq b_n \) for every positive integer \( n \).

Note - Any dominating series consists automatically of non-negative terms although a dominated series may have negative terms.

Theorem XV. Comparison test for convergence.

If \( |a_n| < c \) where \( n \geq N \) (a positive integer), and if the series \( \sum \frac{c_n}{n} \) converges, then the series \( \sum \frac{a_n}{n} \) converges.

Proof: Since the series \( \sum \frac{c_n}{n} \) converges then given \( \varepsilon > 0 \) there exists an \( N = N(\varepsilon) \) such that for \( n \geq m > N \), \( \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} |a_k| < \varepsilon \) whenever \( m \geq n > N \). Therefore \( \sum \frac{a_n}{n} \) converges.

Theorem XVI. Comparison test for divergence.

If \( a_n > d_n > 0 \) for \( n \geq N \) and if \( \sum d_n \) diverges then \( \sum a_n \) diverges.

Proof: This is a consequence of Theorem XV. i.e. if \( \sum a_n \) converges, then \( \sum d_n \) diverges. This is a contradiction.

Examples:

1. Consider the series \( \sum \frac{(-1)^{n-1}}{n^2 + n} \).

   Observe that \( \left| \frac{(-1)^{n-1}}{n^2 + n} \right| \leq \frac{1}{n^2} \) for every \( n \). Hence, by

\[ \text{Ibid., p. 212.} \]
Theorem XV, the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 + n} \) converges since the series \( \sum_{n=1}^{\infty} \frac{1}{n^2 + n} \) converges.

2. Consider the series \( \sum_{n=0}^{\infty} \frac{1}{3n+5} \).

Observe that \( \frac{1}{3n+5} > \frac{1}{4n} > 0 \) where \( n \geq 5 \). Hence, by Theorem XVI, the series \( \sum_{n=0}^{\infty} \frac{1}{3n+5} \) diverges since the series \( \sum_{n=1}^{\infty} \frac{1}{4n} = \sum_{n=1}^{\infty} \frac{a_n}{4} \) diverges.

Theorem XVII. The Root Test.

Consider the series \( \sum_{n=1}^{\infty} a_n \). Set \( b = \lim_{n \to \infty} \sqrt[n]{a_n} \).

If (a) \( b < 1 \), \( \sum_{n=1}^{\infty} a_n \) converges;

(b) \( b > 1 \), \( \sum_{n=1}^{\infty} a_n \) diverges;

(c) \( b = 1 \) no information is given.

Proof: Suppose \( b < 1 \). Let \( \beta \) be such that \( 0 < \beta < 1 \).

Then there exists an index \( M(\beta) \) such that \( \sqrt[n]{a_n} < \beta \) for all \( n > M \). Hence \( a_n < \beta^n \) for all \( n > M \). Therefore \( \sum_{n=M}^{\infty} a_n \) converges since \( \sum_{n=1}^{\infty} \beta^n \) converges.

Suppose \( b > 1 \). Let \( \alpha \) be such that \( 1 < \alpha < b \). Then there exists a subsequence \( \{a_{n_k}\} \) of the sequence \( \{a_n\} \) and a \( M(\alpha) \) such that \( \sqrt[n]{a_{n_k}} > \alpha \) for all \( n_k > N \). Hence \( |a_{n_k}| > \alpha^{n_k} \) and \( a_{n_k} \not\to 0 \).

If \( b = 1 \) it is sufficient to consider the series \( \sum_{n=1}^{\infty} \frac{1}{n} \).
and \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \]. But observe that \( \lim_{n \to \infty} \frac{n}{\sqrt{n}} = \lim_{n \to \infty} \frac{n}{\sqrt{n}} = 1 \).

Therefore no conclusion can be drawn if \( c = 1 \).

Theorem XVIII. The Ratio Test.

Consider the series \( \sum_{n=1}^{\infty} a_n \). Set \( p = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \).

If (a) \( p > 1 \) the series converges;

(b) \( \frac{|a_{n+1}|}{a_n} > 1 \) for \( n > n_0 \) then the series \( \sum_{n=1}^{\infty} a_n \) diverges;

(c) \( \lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} \leq 1 \) then no conclusion can be drawn.

Proof: (a) Let \( \beta \) be such that \( 0 < \beta < 1 \). Then for \( p > 1 \) there exists an \( N = N(\epsilon) \) such that \( \frac{|a_{n+1}|}{a_n} < \beta \) for \( n > N \).

i.e. \( |a_{n+1}| < \beta |a_n| \) so that \( |a_{n+2}| < \beta |a_{n+1}| < \beta^2 |a_n| \).

Hence \( |a_{n+1}| < \beta^{-n} |a_n| \) for \( n > N \).

Therefore, by Theorem XV the series \( \sum_{n=1}^{\infty} a_n \) converges.

(b) \( \frac{|a_{n+1}|}{a_n} \geq 1 \) for all \( n > N_0 \) implies \( a_n \to 0 \) as \( n \to \infty \).

HENCE THE SERIES IS DIVERGENT.

(c) Proof of (c) follows again by considering the series

\[ \sum_{n=1}^{\infty} \frac{1}{n} \] and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).

Observe that \( \lim_{n \to \infty} \frac{h}{h+1} = \lim_{n \to \infty} \frac{h^2}{h+1} = 1 \).

But the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges and the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges. Hence no conclusion can be drawn.
Examples:

1. Consider the series \( \sum_{n=1}^{\infty} \frac{\eta}{n^2} \).

Observe that \( \lim_{n \to \infty} \frac{\eta}{n^2} = \frac{\eta}{1} \leq 1 \). Therefore the series \( \sum_{n=1}^{\infty} \frac{\eta}{n^2} \) converges, by the Root test.

2. Consider the series \( \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{2n+1}{3n^2} \right)^n \).

Observe that \( a_n = \frac{2n+1}{3n^2} \) and \( \frac{a_{n+1}}{a_n} \to \frac{2}{3} < 1 \).

Hence this series converges by the Ratio test.

The Number \( \zeta \).

Definition.

Let \( \zeta = \sum_{n=0}^{\infty} \frac{1}{n!} \).

Theorem XIX.

\[ \zeta = \lim_{n \to \infty} (1 + \frac{1}{n}%) \]

Proof: Set \( S_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \).

Set \( t_n = (1 + \frac{1}{n}%) \). Observe that \( t_n = (1 + \frac{1}{n}%) = 1 + 1 + \frac{1}{2!} (1 - \frac{1}{n}) + \frac{1}{3!} (1 - \frac{2}{n}) + \ldots + \frac{1}{n!} (1 - \frac{n-1}{n}) \).

If \( n \leq S_n \), hence \( t_n \leq \zeta \). Therefore \( \lim t_n \leq \zeta \).

If \( n > m \), then \( t_n > \lim t_m \left( 1 + \frac{1}{2!} (1 - \frac{1}{n}) + \ldots + \frac{1}{m!} (1 - \frac{m-1}{n}) \right) \).

Hold \( a \) fixed and let \( n \to \infty \). Then \( \lim t_n = S_n \) so that \( \lim t_n \neq \zeta \), thus \( \lim t_n \leq \zeta \leq \lim t_n \).

Hence \( \lim t_n = \lim (1 + \frac{1}{n}) \).

Q. E. D.
Theorem XX.

\( \mathfrak{c} \) is irrational.

Proof: Suppose \( \mathfrak{c} \) is rational. That is, \( \mathfrak{c} = \frac{p}{q} \) where \( p \) and \( q \) are integers greater than zero. Consider \( \frac{1}{\mathfrak{c}n} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots \).

Hence we have \( \frac{1}{\mathfrak{c}n} < \frac{1}{n!} \). Now, making use of this inequality, observe that \( \frac{1}{\mathfrak{c}q} < \frac{1}{q!} \). That is \( q!c - q! < \frac{1}{q!} \).

But \( q!c = q! \cdot \frac{p}{q} = (q-1)! \cdot \frac{p}{q} \) is an integer, and \( q! < \frac{1}{q!} (1 + \frac{1}{2} + \cdots + \frac{1}{q}) \) is also an integer. Hence \( q!c - q! < \frac{1}{q!} \) is an integer which is impossible. Q.E.D.

Alternating Series.

Definition.

Suppose an \( a_n > 0 \) where \( n = 1, 2, 3, \ldots \). Then the series \( \sum (-1)^{n-1} a_n \) is called an alternating series.

Theorem XXI - The series \( \sum (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots \) converges if;

(i) \( a_{n+1} < a_n \).

(ii) \( \lim_{n \to \infty} a_n = 0 \).

Proof: Consider \( S_{2n} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots \).

Observe that since \( a_n > a_{n+1} \) for all \( n \), then \( S_{2n} > 0 \).

Hence \( \lim_{n \to \infty} S_n > 0 \) if it exists. Consider \( a_n - (a_2 - a_3) - (a_4 - a_5) \).
This means that the partial sums are bounded. In particular, the \( \lim_{n \to \infty} S_{n+1} \leq q_1 \).

Note, \( S_{2n+1} = S_{2n} + q_{2n+1} \), i.e., \( S_{2n+1} = \lim a_n > S_{2n} \)
and hence \( 0 \leq S_{2n} \leq S_{2n+1} < q_1 \). By Theorem IIb we have \( \lim_{n \to \infty} (S_{2n+1} - S_{2n}) \)
\[ \lim a_{n+1} = 0. \text{ That is } \lim S_{2n+1} = \lim S_{2n}. \]
Therefore \( \sum_{n=1}^{\infty} (1)^{n-1} a_n \) converges. We set \( \sum (-1)^{n-1} a_n = S \) and claim that \( S - S_n \leq q_{n+1} \).

Note that \( S - S_{n-1} = (-1)^n [a_{n+1} - a_{n+2} + \ldots] \).
However, from above we have that the sum \( a_{n+1} - a_{n+2} + a_{n+3} - \ldots \)
\( < q_{n+1} \). Therefore \( |S - S_n| < q_{n+1} \) for all \( n \).

**Absolute Convergence**

The series \( \sum_{n=1}^{\infty} a_n \) converges absolutely if and only if the
series \( \sum_{n=1}^{\infty} |a_n| \) converges. If the series \( \sum_{n=1}^{\infty} |a_n| \) converges, but does not converge absolutely, then the series \( \sum_{n=1}^{\infty} a_n \)
is said to converge conditionally.

**Theorem XXII.**

Absolute convergence implies convergence.

Proof: Assume the series \( \sum a_n \) converges. Note that a series
\( \sum a_n \) converges if and only if for \( \varepsilon > 0 \) there exists \( N = N(\varepsilon) \)
such that \( \left| \sum_{k=m}^{n} a_k \right| < \varepsilon \) for all \( n > m > N \). Therefore for \( \varepsilon > 0 \)
there exists \( N = N(\varepsilon) \) such that \( \sum_{k=m}^{n} |a_k| < \varepsilon \) for all \( n > m > N \).
Hence \( \left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| < \varepsilon \) for all \( n > m > N \).

Q.E.D.
Domain of Convergence.

Definition.

Let \( \{c_n\} \) be a sequence of complex numbers. Let \( z \) be a complex number. Then the series \( \sum_{n=0}^{\infty} c_n z^n \) is a power series and converges on diverges accordingly as \( z \) belongs or does not belong to the domain of convergence.

Definition.

Domain of convergence implies the set of all \( z \) for which the series \( \sum_{n=0}^{\infty} c_n z^n \) converges. The domain of convergence of a power series is a circle in the interior of which the series converges. The series diverges in the exterior of the circle. It is not easy to characterize what happens on the boundary of the circle; therefore, to include all cases, we consider the whole finite plane as a circle of finite radius and a point as a circle of radius zero.

Theorem XXIII.

Set \( g = \lim_{k \to \infty} \sqrt[k]{|a_k|} \) Put \( R = \frac{1}{g} \) For \( g = 0 \), \( R = +\infty \) For \( g = +\infty \), \( R = 0 \). If \( |z| < R \), the series converges. If \( |z| > R \) the series diverges.

Proof: Put \( a_n = c_n z^n \). By Theorem XVII we have

\[
\lim \sqrt[n]{|a_n|} = \lim \sqrt[n]{|c_n| \cdot |z^n|} = |z| \cdot \lim \sqrt[n]{|c_n|} = \frac{|z|}{R}
\]

Hence the results. Q.E.D.
Examples:

1. Consider the series \( \sum \frac{z^n}{n} \). Observe that \( a_n = \frac{z^n}{n} \)

and \( a_{n+1} = \frac{z^{n+1}}{(n+1)!} \). Then \( \frac{a_{n+1}}{a_n} = \frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n} = \frac{z^n}{n+1} \).

Thus we have that \( \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{z^n}{n+1} \right| \to 0 \) as \( n \to \infty \), \( R = +\infty \).

Hence the series converges for all \( |z| < \infty \).

2. Consider the series \( \sum z^n \). Observe that \( \lim n^{\frac{1}{n}} = +\infty \)

therefore the series converges for \( z = 0 \).

3. Consider the series \( \sum \frac{z^n}{n} \). Observe that \( R=1 \), therefore the series converges for all \( |z| < 1 \) and for no points on the boundary.

4. Consider the series \( \sum \frac{z^n}{n^2} \). Observe that \( R=1 \), i.e., \( \lim n^{-\frac{1}{n}} = 1 \)

\( \sqrt[n]{n} = \left( \frac{1}{n} \right)^{\frac{1}{n}} = e^{\frac{\ln 1}{n}} = e^{-\frac{n}{\ln n}} \to e^0 = 1 \)

as \( n \to \infty \). Thus the domain of convergence is \( |z| < 1 \).

The series does not converge at \( z=1 \) but converges on all other points of the boundary.

5. Consider the series \( \sum \frac{z^n}{n^2} \). Observe that \( R=1 \), therefore the circle of convergence is \( |z| < 1 \). But hence the series converges at all points of the boundary since \( \left| \frac{z}{n^2} \right| \leq \frac{1}{n^2} \) for all \( z \in |z| = 1 \).

Partial Summation.

**Theorem XXIV.**

Let \( \{a_n\} \) and \( \{b_n\} \) be sequences of real numbers. Set

\[ A = \sum_{n=0}^{\infty} a_n \quad \text{for} \ n \geq 0 \quad \text{and set} \quad A_{-1} = 0. \]

Let \( 0 \leq p \leq 1 \). Then
Abel's sum, \[ \sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A(b_{n+1} - b_n) + A_q b_q - A_{p-1} b_p. \]

Proof: Observe that \[ \sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n \]
\[ = \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n, \]
and, \[ \sum_{n=p}^{q} A_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n \]
\[ = \sum_{n=p}^{q-1} A_n b_n - \sum_{n=p}^{q-1} A_{n-1} b_n, \]
where \( n' = n - 1 \) and \( n = n' + 1 \). Since these indices are dummy we may write that the above series \[ \sum_{n=p}^{q} A_n b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n'=p}^{q-1} A_{n'} b_{n'+1}, \]
\[ = \sum_{n=p}^{q} A_n b_n - A_{q-1} b_q - [A_{p-1} b_p + \sum_{n=p}^{q-1} A_{n} b_{n+1}]. \]
\[ = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p. \]
Q.E.D.

Theorem XXV.

If (a) \( A_n \) of \( \{a_n\} \) forms a bounded sequence.

(b) \( b_o \geq b_1 \geq b_2 \geq \ldots \).

(c) \( \lim_{n \to \infty} b_n = 0. \)

Then the series \( \sum a_n \) converges.

Proof: Let \( M > 0 \) be such that \( |A_n| \leq M \) for all \( n \).
Let \( N(\varepsilon) \) be given arbitrarily such that \( b_n \leq \varepsilon \). Then observe that for \( q \geq p > N \) we have
\[ \left| \sum_{n=p}^{q} a_n b_n \right| = \left| \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \right| \leq M \left| \sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p \right| = 2Mb_p \leq 2Mb_n \leq \varepsilon. \]

Theorem XXVI.

If (a) \( c_o \geq c_1 \geq c_2 \geq \ldots \).
(b) \( \lim_{n \to \infty} C_n = 0 \)

(c) Radius of convergence of the series \( \sum_{n=0}^{\infty} C_n z^n \) is 1.

Then the series \( \sum_{n=0}^{\infty} C_n z^n \) converges for all \( z \) on \( |z| = 1 \) except possibly at the point \( |z| = 1 \).

Proof: Set \( A_n = z^n \) and \( b_n = C_n \). Then

\[
|A_n| = \left| \sum_{k=0}^{n} a_k \right| = \left| \sum_{k=0}^{n} z^k \right| = \left| \frac{1 - z^{n+1}}{1 - z} \right| < \frac{2}{1 - 2} \quad \text{for all} \ n \quad \text{and}
\]

therefore for all \( z \) on \( |z| = 1 \) except \( z = 1 \). Hence (i) \( a_0 + a_1 + a_2 + \cdots + a_n \) is bounded for all \( n \).

(ii) \( b_0 \geq b_1 \geq b_2 \geq \cdots \) (iii) \( \lim_{n \to \infty} b_n = 0 \).

Therefore, by Theorem XXV, \( \sum_{n=0}^{\infty} C_n z^n \) converges for all \( z \) on \( |z| = 1 \) except possibly \( z = 1 \).

Addition of convergent series.

Theorem XXVII.

If the series \( \sum_{n=0}^{\infty} a_n \) and the series \( \sum_{n=0}^{\infty} b_n \) are convergent series and if \( \alpha \) and \( \beta \) are any constants, then the series

\[
\sum_{n=0}^{\infty} \left( \alpha a_n + \beta b_n \right)
\]

converges.

Proof: Set \( C_n = \alpha a_n + \beta b_n \). Then the series \( \sum_{k=0}^{n} C_k \)

\[
= \sum_{k=0}^{n} (\alpha a_k + \beta b_k) = \alpha \left( \sum_{k=0}^{n} a_k \right) + \beta \left( \sum_{k=0}^{n} b_k \right)
\]

But since \( \lim_{n \to \infty} \sum_{k=0}^{n} a_k \) and \( \lim_{n \to \infty} \sum_{k=0}^{n} b_k \) exist and are finite, we have that \( \lim_{n \to \infty} \sum_{k=0}^{n} C_k \) exists and is finite, i.e.,

\[
\sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)
\]

converges. If we set \( \lim_{n \to \infty} \sum_{k=0}^{n} a_k = A \),

and \( \lim_{n \to \infty} \sum_{k=0}^{n} b_k = B \) we have that \( \lim_{n \to \infty} \sum_{k=0}^{n} C_k = \alpha A + \beta B \).
i.e., \[ \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n) = \alpha A + \beta B. \]

Multiplication of convergent series.

Definition.

Consider the series \( \sum_{n=0}^{\infty} a_n \) and \( \sum_{n=0}^{\infty} b_n \). Set \( c_n = \sum_{k=0}^{n} a_k b_{n-k} \).

Then the product of \( \sum_{n=0}^{\infty} a_n \) and \( \sum_{n=0}^{\infty} b_n \) is the series \( \sum_{n=0}^{\infty} c_n \).

The product of \( c_n \) is defined because if we multiply the two series together we have \( (\sum a_n z^n)(\sum b_n z^n) \)

\[ = (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) \]

\[ = a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + b_1 a_2) z^2 + \cdots \]

\[ = \sum_{n=0}^{\infty} \left[ \sum_{k=1}^{n} a_k b_{n-k} \right] = \sum_{n=0}^{\infty} c_n. \]

But the convergence of two series is not sufficient for the convergence of their product. For example, consider the series

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots \]

Then \( \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \right) \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \right) = 1 - \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) + \left( \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) + \cdots \)

so that \( c_n = \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(n+1)}} \) but \( (n-k+1)(k+1) \)

\[ = (n+1)^2 - (k+1)^2 = (n+1)^2. \]

Hence \( |c_n| \geq \sum_{n=0}^{\infty} \frac{1}{n+2} = \frac{2(n+1)}{n+2} \rightarrow 2, \) so that the \( n^{th} \) term of the series \( \sum c_n \) does not tend to Zero. Therefore the series \( \sum c_n \) diverges.

Theorem XXVIII.

If the series \( a_n \) is absolutely convergent,

(b) \( \sum_{n=0}^{\infty} a_n = A \).
(c) \[ \sum_{n=0}^{\infty} b_n = B, \]
then \[ \sum_{n=0}^{\infty} C_n = A \cdot B \]
where \[ C_n = \sum_{k=0}^{n} a_k - b_{n-k}. \]

Proof: Set \( b_n = b_0 + b_1 + b_2 + \cdots + b_n \). Set \( A_n = B - B_n \).
Then observe that \( C_n = a_0 b_0 + (a_0 b_1 + a_1 b_{n-1}) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) = a_0 (B - B_n) + a_1 (B - B_{n-1}) + \cdots + a_n (B - B_0) \).

Note that \( \lim_{n \to \infty} A_n B = A \cdot B \). Therefore it is sufficient to show that \( \Gamma_n = \sum a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0 \to 0 \) as \( n \to +\infty \).

Since the series \( \sum_{n=0}^{\infty} a_n \) is absolutely convergent, \( \sum_{n=0}^{\infty} |a_n| = \alpha \).

Therefore, given \( \varepsilon > 0 \) there exists an \( N = N(\varepsilon) \) such that \( |\Gamma_n| < \varepsilon \) for all \( n > N \). Hence, for all \( n > N \) we have

\[ |\Gamma_n| = |a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0| \leq |a_0| |\beta_n| + |a_1| |\beta_{n-1}| + \cdots + |a_n| |\beta_0| \leq \alpha. \]

Fix \( N \) and let \( n \to \infty \). Then since \( \lim_{n \to \infty} a_n = 0 \) we have \( \lim_{n \to \infty} \sup_{K} |\beta_K| = 0 \). But \( \varepsilon \) is arbitrary.

Hence \( \Gamma_n \to 0 \) as \( n \to +\infty \). Q.E.D.

Theorem XXIX. (Abel's Theorem)

If the series \( \sum_{n=0}^{\infty} a_n = A, \sum_{n=0}^{\infty} b_n = B, \sum_{n=0}^{\infty} c_n \)
converge where \( c_n = \sum_{k=0}^{n} a_k b_{n-k} \), then \( \sum_{n=0}^{\infty} c_n = A \cdot B. \)
Rearrangements.

Definition.

Let \( \{N_k\} \) be a sequence in which for each \( k \) one and only one positive integer appears. i.e., \( \{N_k\} \) is the set of non-negative integers in a different order. Another way of saying this is:
the series \( \{N_k\} \) is a 1-1 function which maps the non-negative integers onto the non-negative integers.

Consider the series \( \sum a_n \). Set \( a'_k = a_{n_k} \). Then the series \( \sum a'_k = \sum a_n \) is called a rearrangement of \( \sum a_n \). In general however, we cannot conclude necessarily that if \( \sum a_n \) converges then \( \sum a'_k = \sum a_k \). Consider the series, \( \sum_{n=1}^{\infty} \left( (-1)^{n-1} \frac{1}{n} \right) \) and the series \( \sum_{n=1}^{\infty} \left[ \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right] = 1 - \frac{\sqrt{2}}{2} + \frac{1}{2} + \frac{1}{\sqrt{2}} \)

\[ - \frac{1}{4} + \frac{1}{7} + \frac{1}{11} - \frac{1}{6} + \cdots \]

Observe that if \( S = \sum_{n=1}^{\infty} \left( (-1)^{n-1} \frac{1}{n} \right) \) then \( (1 - \frac{1}{2} + \frac{1}{3}) = \frac{5}{6} > S \) and if \( S' = \sum_{n=1}^{\infty} \left[ \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right] \), then \( (1 + \frac{1}{2} - \frac{1}{3}) = S' \) (since \( \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} > 0 \) for all \( n \)).

Note that, \( \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} > 0 \). Therefore we have that \( S < S' \) and hence the assertion made above.

**Definition.**

If every rearrangement of a series converges to the same sum, then that series is said to be unconditionally convergent.

**Theorem XXX.**

If the series \( \sum a_n \) converges absolutely then the series \( \sum a_n \) is unconditionally convergent.

**Proof:** (a) Suppose \( a_n \geq 0 \). Consider the series \( \sum a'_k = \sum a_{n_k} \) as any rearrangement of \( \sum a_n \). Set \( S'_k = a_{n_1} + a_{n_2} + a_{n_3} + \cdots + a_{n_k} \). Set \( S_k = a_1 + a_2 + a_3 + \cdots + a_k \). Let \( N = \max \{ n_1, n_2, \ldots, n_k \} \). Then \( S'_k \leq S_N = a_{n_1} + a_{n_2} + \cdots + a_{n_k} \). Thus \( S'_k \leq S \leq \sum_{n=1}^{\infty} a_n \) for every \( k \). Hence \( \lim_{n \to \infty} S_n \) exists and is less than or equal to \( S_a \).
Call this limit $s'$. Therefore $s' \leq s$. Equivalently we can consider $\sum a_n$ as a rearrangement of $\sum a_{nk}$. Then following the same type of reasoning as was exhibited above we obtain $s \leq s'$; Therefore $s = s'$.

(b) Suppose $\sum a_n$ converges absolutely. Then $\sum |a_n|$ converges. Set $p_n = \frac{|a_n|}{2} + a_n \geq 0$ and $q_n = \frac{|a_n| - a_n}{2} \geq 0$, then $p, q \geq 0$ for all $n$. Moreover, the series $\sum p_n$ and $\sum q_n$ converge.

By part (a), any rearrangement of the series $\sum a_n$ and $\sum q_n$ say, $\sum p'_n$ and $\sum q'_n$ converge to the same sums as $\sum p_n$ and $\sum q_n$ respectively. i.e. If $p = \sum p_n$ and $Q = \sum q_n$, then $\sum p'_n = p$ and $\sum q'_n = Q$. But $p'_n = \frac{1}{2} |a'_n| + a_n$ and $q'_n = \frac{1}{2} |a'_n| - a_n$, so that $p'_n - q'_n = a'_n$ and $p'_n + q'_n = |a'_n|$. But $\sum a'_n = \sum p'_n - \sum q'_n = P - Q$ and $\sum |a'_n| = \sum p'_n + \sum q'_n = P + Q$. Equivalently $\sum a_n = p - q$ and $\sum |a_n| = p + q$. Hence $\sum |a_n| = \sum |a'_n|$ and $\sum a_n = \sum a'_n$. Q.E.D.
CHAPTER III

INTEGRATION

Let $f$ be a real valued function defined on the closed interval $[a, b]$. Then the set of points, $x_0, x_1, x_2, x_3, \ldots, x_n,$ where $x_0 = a$ and $x_n = b$ is called a net and is denoted by $N$.

$[x_{i-1}, x_i]$ is a sub interval of $[a, b]$. We denote the length of $[x_{i-1}, x_i]$ by $A(x_{i-1}, x_i) = x_i - x_{i-1}$, where $i = 1, 2, 3, \ldots, n$. The norm of $N$ is defined to be $\max A(x_{i-1}, x_i)$. Let $\xi \in [x_{i-1}, x_i]$. Then consider the sum:

$$\sum_{i=1}^{n} f(\xi_i) A(x_{i-1}, x_i) = \sum_{i=1}^{n} f(\xi_i) (x_i - x_{i-1}) = \sum_{i=1}^{n} f(\xi_i) A(x_{i-1}, x_i) + f(\xi_1) (x_1 - x_0) + \ldots + f(\xi_n) (x_n - x_{n-1}).$$

Definition.

Let $N$ be an arbitrary net for the interval $[a, b]$. Then the function $f$ is said to be integrable, (in the Riemann Sense) if and only if:

$$\lim_{\|N\| \to 0} \sum_{i=1}^{n} f(\xi_i) A(x_{i-1}, x_i)$$

exists and is finite. This limit (if it exists) is called the definite integral of $f(x)$ over the interval $[a, b]$ and is denoted by:

$$\int_{a}^{b} f(x) \, dx = \lim_{\|N\| \to 0} \sum_{i=1}^{n} f(\xi_i) A(x_{i-1}, x_i).$$

The definite integral is a special sum. It is an infinite sum. The equality above is a transformation of quantity into quality. A quantity of sums, on the right, builds up a sum which is represented on the left. The left hand side is and yet is not a sum.

Immediate consequences of this Definition.
1. \( \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \),

2. \( \int_{a}^{a} f(x) \, dx = 0 \).

Example. Suppose \( f(x) = x \). Find \( \int_{a}^{b} f(x) \, dx \).

Consider; \( J' = \sum_{i=1}^{n} x_{i-1} (x_{i} - x_{i-1}) \) and \( J'' = \sum_{i=1}^{n} x_{i} (x_{i} - x_{i-1}) \).

Then \( \lim_{|N| \to 0} (J' + J'') = \int_{a}^{b} x \, dx \).

\[
\begin{align*}
\sum_{i=1}^{n} x_{i} (x_{i} - x_{i-1}) &= \sum_{i=1}^{n} x_{i} (x_{i} - x_{i-1}) = \sum_{i=1}^{n} (x_{i}^{2} - x_{i-1}^{2}) \\
&= (x_{1}^{2} - x_{0}^{2}) + (x_{2}^{2} - x_{1}^{2}) + \cdots + (x_{n}^{2} - x_{n-1}^{2}) \\
&= x_{n}^{2} - x_{0}^{2}.
\end{align*}
\]

Therefore \( \int_{a}^{b} x \, dx = \frac{1}{2} (x_{n}^{2} - x_{0}^{2}) \).

Properties of the Riemann Integral.

Theorem I. If \( \lim_{|N| \to 0} \sum_{i=1}^{n} f(\xi_{i}) \Delta x_{i} \) exists, then it is unique.

Proof: Suppose \( \lim_{|N| \to 0} \sum_{i=1}^{n} f(\xi_{i}) \Delta x_{i} = I \) and \( \lim_{|N| \to 0} \sum_{i=1}^{n} f(\xi_{i}) \Delta x_{i} = J \).

Suppose \( I \neq J \). Then set \( \varepsilon = \frac{1}{2} |I - J| \). By our assumption, there exists a \( \delta_{1} = \delta_{1}(\varepsilon) \) such that \( |\sum_{i=1}^{n} f(\xi_{i}) \Delta x_{i} - I| < \frac{\varepsilon}{2} \) for \( |N| < \delta_{1} \), and there exists a \( \delta_{2} = \delta_{2}(\varepsilon) \) such that \( |\sum_{i=1}^{n} f(\xi_{i}) \Delta x_{i} - J| < \frac{\varepsilon}{2} \) for \( |N| < \delta_{2} \). Let \( \delta = \min \{ \delta_{1}, \delta_{2} \} \). Then \( 2\varepsilon = |I - J| = |J - \sum_{i=1}^{n} f(\xi_{i}) \Delta x_{i} + \sum_{i=1}^{n} f(\xi_{i}) \Delta x_{i} - I| \leq |\sum_{i=1}^{n} f(\xi_{i}) \Delta x_{i} - J| + |\sum_{i=1}^{n} f(\xi_{i}) \Delta x_{i} - I| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \) for \( |N| < \delta \), which implies \( I = J \). Therefore \( I = J \). Q.E.D.

Theorem II. If \( f \) and \( g \) are integrable on \([a, b]\) and if \( f(x) \leq g(x) \) on \([a, b]\), then \( \int_{a}^{b} f(x) \, dx \leq \int_{a}^{b} g(x) \, dx \).

Proof: Set \( I = \int_{a}^{b} f(x) \, dx \) and \( J = \int_{a}^{b} g(x) \, dx \).

Suppose \( I > J \). Set \( \varepsilon = \frac{1}{2} (I - J) \). Then \( I - \varepsilon = J + \varepsilon \). There exists a \( \delta_{1} \) such that \( |\sum_{i=1}^{n} f(\xi_{i}) \Delta x_{i} - I| < \varepsilon \) for \( |N| < \delta_{1} \). Also, there
exists a $\delta_2$ such that $|\sum q(x_i)dx_i - J| < \epsilon$ for $|N| < \delta$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\sum q(x_i)dx_i < J + \epsilon < \sum f(x_i)dx_i$ for $|N| < \delta$. This is an inequality which is incompatible with $f(x) \leq q(x)$ on $[a, b]$. Q.E.D.

Theorem III. If $f$ is integrable on $[a, b]$ and $k$ is any constant, then $kf$ is integrable on $[a, b]$ and $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.

Proof: Let $f$ be given. There exists a $\delta = \delta(x)$ such that $|\sum f(x_i)dx_i - \int_a^b f(x)dx| < \frac{\epsilon}{|k|}$ for $|N| < \delta$. Observe that,

$|\sum k f(x_i)dx_i - k\int_a^b f(x)dx| = |k| |\sum f(x_i)dx_i - \int_a^b f(x)dx| < |k| \frac{\epsilon}{|k|} = \epsilon$ for $|N| < \delta$. Hence $\int_a^b kf(x)dx = k\int_a^b f(x)dx$. Q.E.D.

Theorem IV. If $f$ and $q$ are integrable on $[a, b]$ then $f \neq q$ is integrable on $[a, b]$, and $\int_a^b [f(x) + q(x)]dx = \int_a^b f(x)dx + \int_a^b q(x)dx$.

Proof: Set $I = \int_a^b f(x)dx$ and $J = \int_a^b q(x)dx$. Then for $\epsilon > 0$ there exists a $\delta_1 = \delta_1(\epsilon)$ such that $|\sum f(x_i)dx_i - I| < \frac{\epsilon}{2}$ for $|N| < \delta_1$. Also, there exists a $\delta_2 = \delta_2(\epsilon)$ such that $|\sum q(x_i)dx_i - J| < \frac{\epsilon}{2}$ for $|N| < \delta_2$. Hence $|\sum f(x_i)dx_i - I + \sum q(x_i)dx_i - J| < \epsilon$ for $|N| < \delta$. Q.E.D.

Theorem V. If $f$ and $q$ are integrable on $[a, b]$ and $q(x) \geq f(x)$ on $[a, b]$ except possibly at a finite number of points on $[a, b]$, then $\int_a^b q(x)dx \geq \int_a^b f(x)dx$.

Proof: Let $I = \int_a^b f(x)dx$ and $J = \int_a^b q(x)dx$. Assume $I > J$ and set $\epsilon = \frac{1}{2} (I - J)$. Then there exists a positive number $\delta$ so small that for any net $N$ of norm less than $\delta$ and for any choice of points $x_1, x_2, x_3, \ldots, x_n$, the following inequalities hold simultaneously;

$\sum [q(x_i)dx_i - J + \epsilon] = I - \epsilon < \sum f(x_i)dx_i$. But
this implies an inequality inconsistent with the assumed inequality that \( f(x) \leq g(x) \).

**Theorem VI.** If \( a < b < c \) and if \( f \) is integrable on \([a, b]\) and \( g \) is integrable in \([b, c]\), then \( f \) is integrable on \([a, c]\) and

\[
\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx.
\]

**Proof:** Let \( I = \int_a^b f(x) \, dx \) and \( J = \int_b^c f(x) \, dx \). Then given \( \varepsilon > 0 \), there exists a \( \delta_1 = \delta_1(\varepsilon) \) such that

\[
\left| \sum_{i=1}^n f(\xi_i) \Delta x_i - I \right| < \frac{\varepsilon}{2}
\]

for \( |N| < \delta_1 \). Also, there exists a \( \delta_2 = \delta_2(\varepsilon) \) such that

\[
\left| \sum_{i=1}^n f(\eta_i) \Delta x_i - J \right| < \frac{\varepsilon}{2}
\]

for \( |N| < \delta_2 \). Let \( \delta = \min \{ \delta_1, \delta_2 \} \). Set \( f(\xi_i) = f(\xi_i) + f(\eta_i) \) where \( \xi_i \in [x_{i-1}, x_i] \) for \( |N| < \delta \). Observe that

\[
\left| \sum_{i=1}^n f(\xi_i) \Delta x_i - (I + J) \right| = \left| \sum_{i=1}^n [f(\xi_i) + f(\eta_i)] \Delta x_i - (I + J) \right|
\]

\[
= \left| \sum_{i=1}^n f(\xi_i) \Delta x_i - I \right| + \left| \sum_{i=1}^n f(\eta_i) \Delta x_i - J \right| 
\]

\[
\leq \left| \sum_{i=1}^n f(\xi_i) \Delta x_i - I \right| + \left| \sum_{i=1}^n f(\eta_i) \Delta x_i - J \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]

for \( |N| < \delta \). Thus

\[
\lim_{|N| \to 0} \sum_{i=1}^n f(\xi_i) \Delta x_i = I + J = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx.
\]

Since \( I + J \) is a finite sum, then \( \int_a^c f(x) \, dx \) exists and

\[
\int_a^c f(x) \, dx = I + J = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx. \quad \text{Q.E.D.}
\]
Theorem VII. Any function continuous on a closed interval is integrable there.

Proof: Since \( f(x) \) is continuous on the closed interval \([a, b]\) then it is uniformly continuous there. Assume \( b > a \). Then, given \( \varepsilon > 0 \) there exists a positive number \( \delta \) such that

\[ |x' - x''| < \delta \quad \text{implies} \quad |f(t) - f(x''')| < \frac{\varepsilon}{b-a}. \]

Let \( \delta \) be any net of norm less than \( \delta \), and, let \( \sigma_i \) and \( \tau_i \) for \( i = 1, 2, \ldots, n \) be defined as the minimum and maximum values respectively of \( f(x) \) on the (sub-interval) \([a_i, a_i']\). If \( x_{i'} \) and \( x_{i''} \) are points of \([a_i, a_i']\) such that \( f(x_{i'}) = \sigma_i \) and \( f(x_{i''}) = \tau_i \) where \( i = 1, 2, \ldots, n \) and if the step functions \( \sigma(x) \) and \( \tau(x) \) are defined to have the values \( \sigma_i \) and \( \tau_i \) respectively for

\[ a_{i_1} < x < a_{i}. \]

where \( i = 1, 2, \ldots, n \), then \( \sigma(x) \leq f(x) \leq \tau(x) \) and

\[
\int_a^b [\tau(x) - \sigma(x)] \, dx = \sum_{i=1}^n (\tau_i - \sigma_i) a_i \varepsilon_i = \sum_{i=1}^n |f(x_i)|
\]

\[ - f(x_{i''}) \int_{a_i}^{x_i} < \frac{\varepsilon}{b-a} \sum_{i=1}^n (x_{i'} - x_{i''}) = \frac{\varepsilon}{b-a} (x_n - x_0) = \frac{\varepsilon}{b-a} (b-a) = \varepsilon. \]

Q.E.D.

Theorem VIII. If the function \( f \) is bounded and continuous almost everywhere on \([a, b]\) then \( f \) is integrable there.

Proof: Let \( f \) be continuous on \([a, b]\) except for a finite number of discontinuities \( \kappa \) at \( x = c \). In any net over \([a, b]\), \( c \) is a point of at most two sub-intervals. i.e. \( c = x_i \) where \( x_i \) is a net point. The oscillation in these intervals is \( \leq M - m \) for a net \( |N| \) of mesh \( \delta_i \). Their contribution to \( \delta - \delta \) is \( \delta = R(M - m) \delta_i \). Let \( \delta_i < \delta \)

where \( \delta_i \) is so small that \( \delta = R(M - m) \delta_i < \frac{\varepsilon}{\delta(b-a)} \).
in all sub-intervals in which \( f \) is continuous. Thus we have
\[
\frac{1}{4} \varepsilon + \frac{\ell R(b-a)}{4 R(b-a)} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Hence \( f \) is integrable in \([a, b]\). Q.E.D.

Theorem IX. A function defined and monotonic on a closed interval is integrable there.

Proof: Assume for definiteness that \( f(x) \) is monotonically increasing on \([a, b]\) and for a given net \( N \) define the step functions \( \sigma(x) = f(a_i) \) for \( a_i < x < a_i \), where \( i = 1, 2, \ldots, n \) and \( \tau(x) = f(a) \). Then
\[
\sigma(x) \leq f(x) \leq \tau(x) \quad \text{and} \quad \int_a^b [\tau(x) - \sigma(x)] \, dx = \sum_{i=1}^n [f(a_i) - f(a_{i-1})] \Delta x \leq |N| \sum_{i=1}^n [f(a_i) - f(a_{i-1})] = |N| (f(b) - f(a)).
\]

By a previous Theorem (\#IV), it was shown that
\[
\int_a^b [\tau(x) - \sigma(x)] \, dx = \int_a^b f(x) \, dx \quad \text{Hence} \quad \int_a^b f(x) \, dx \leq |N| [f(b) - f(a)].
\]
which shows that \( \lim_{|N| \to 0} \sum_{i=1}^n f(\xi_i) \Delta x_i \) exists, i.e., it is finite since this limit is less than \( |N| \cdot [f(b) - f(a)] \) which is a finite quality. Q.E.D.

Theorem X. First mean value theorem for Integrals. If \( f \) is continuous on a closed interval \([a, b]\) then there exists a point \( \xi \in [a, b] \) such that
\[
\int_a^b f(x) \, dx = f(\xi) (b-a).
\]

Proof: Let \( m \) and \( s \) denote the minimum and maximum values of \( f(x) \) on \([a, b]\). Then consider the sum \( f(x_i) \Delta x_1 + \cdots + f(x_n) \Delta x_n \).

Then \( m \leq f(x_i) \leq s \) where \( i = 1, 2, 3, \ldots \) and \( m \sum_{i=1}^n \Delta x_i \leq \sum_{i=1}^n f(x_i) \Delta x_i \leq s \sum_{i=1}^n \Delta x_i \).

But \( \sum_{i=1}^n \Delta x_i = (b-a) \). Therefore the sum \( f(x') \Delta x_1 + f(x_2) \Delta x_2 + \cdots + f(x_n) \Delta x_n \)
lies between \( m(b-a) \) and \( s(b-a) \). Since by definition of a definite integral as a limit of a sum we have \( m(b-a) \leq \int_a^b f(x) \, dx \leq s(b-a) \) then \( m \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq s \). By setting \( N = \frac{1}{b-a} \int_a^b f(x) \, dx \) we see that for \( a \leq \xi \leq b \), \( f(\xi) \) can take on the value \( N \) since \( m \leq N \leq s \).

Therefore \( f(\xi)(b-a) = \int_a^b f(x) \, dx \). Q.E.D.

Definition. \( f(\xi) = g(\xi) \) almost everywhere on \([a, b]\) if and only if \( f(\xi) = g(\xi) \) for points on \([a, b]\) which have length zero.

Definition. P.P. means that a condition holds almost everywhere in \([a, b]\) if and only if it fails to hold on a set of points in \([a, b]\) of length zero.

Proof: Let \( S \) be the set in question. Then, since \( S \) is countable, it can be enumerated in the following manner:
\( S_1, S_2, S_3, \ldots \)

Let \( \epsilon > 0 \) be given and let \( G = \{ \xi \mid \text{dist}(\xi, S) < \frac{\epsilon}{2} \} \). Note, \( G = \bigcup_{i=1}^{\infty} G_i \) is an open covering for the set \( S \).

\[
\ell(G) = \sum_{i=1}^{\infty} \ell(G_i) < \sum_{i=1}^{\infty} \frac{\epsilon}{2} = \epsilon.
\]

Therefore \( \ell(S) \leq \ell(G) \). But \( \epsilon \) is arbitrary therefore \( \ell(S) = 0 \). Q.E.D.

Theorem XI. If a function is constant on \([a, b]\) and in particular if \( f(\xi) = k \) on \([a, b]\), then \( \int_a^b f(x) \, dx = k(b-a) \).

Proof: Observe that for any \( N \):
\[
\sum f(\xi_i) \Delta \xi_i = k \sum \Delta \xi_i
\]

\[
= k \sum (\xi_i - \xi_{i-1}) = k \left[ X_n - X_0 + X_{n-1} - X_{n-2} + \cdots + X_1 - X_0 \right]
\]

\[
= k \left[ X_n - X_0 \right] = k[b-a] \quad \text{as} \quad \xi \to 0, \quad \text{therefore}
\]

\[
\int_a^b f(x) \, dx = k(b-a). \quad \text{Q.E.D.}
\]
The Riemann–Stieltjes Integral.

Definition. Let \( f \) be bounded on the closed interval \([a, b]\).

Let \( N \) be any net for the interval \([a, b]\). Let \( dx = x_i^--x_{i-1} \),

Let \( m_i = \inf f(x) \) and \( M_i = \sup f(x) \) where \( x_{i-1} \leq x \leq x_i \).

Set \( u(f, N) = \sum_{i=1}^n M_i \Delta x_i \) and \( L(f, N) = \sum_{i=1}^n m_i \Delta x_i \). Then

\[ \int_a^b f(x) \, dx = \inf \{ u(f, N) \} \equiv \text{Riemann Lower Integral} \quad \text{and} \quad \int_a^b f(x) \, dx = \sup \{ l(f, N) \} \equiv \text{Riemann Upper Integral}. \]

If \( \int_a^b f dx = \int_a^b f dx \), then \( f \) is Riemann Integrable on \([a, b]\),
i.e. \( f \in \mathcal{R}_j \), the class of Riemann on \([a, b]\) and \( \int_a^b f dx = \int_a^b f dx \)

\[ \int_a^b f dx \equiv \text{Riemann Integrable on \([a, b]\)}. \]

A necessary condition for the existence of the Riemann integral is boundness and continuity almost everywhere. The Riemann integral is a special case of the Riemann–Stieltjes integral.

Definition. Let \( \alpha \) be monotonic increasing on \([a, b]\). Then since \( \alpha(a) \) and \( \alpha(b) \) and finite values, \( \alpha \) is bounded on \([a, b]\).

Set \( \Delta \alpha_i = (\alpha x_i - \alpha x_{i-1}) \geq 0 \), for \( i = 1, 2, 3, \ldots \). Set \( L(f, N, \alpha) = \sum_{i=1}^n m_i \Delta \alpha_i \), where \( i = 1, 2, 3, \ldots \). Set \( u(f, N, \alpha) = \sum_{i=1}^n M_i \Delta \alpha_i \) where \( i = b, 2, \ldots \). Then \( \int_a^b f d\alpha = \int_a^b f d\alpha \equiv R-S \) upper integral, and \( \int_a^b f d\alpha = \int_a^b f d\alpha \equiv R-S \) lower integral.

If \( \int_a^b \int_a^b f(x) \, dx \), then \( f \) is Riemann–Stieltjes integrable to \( \alpha \) on \([a, b]\). Its common value is denoted by \( \int_a^b f(x) \, d\alpha \).

Let \( \alpha \) be defined by \( \alpha(x) = x \) on \([a, b]\). In this case, the Riemann integral reduces to the Riemann–Stieltjes Integral. Hence \( \alpha \in \)
\( \Delta x_i \) hence the Riemann integral is a special case of the Riemann–Stieltjes Integral.

**Notations**

\[ M_i \equiv \int_{x_{i-1}}^{x_i} f(x) \, dx, \text{ where } x_{i-1} \leq x \leq x_i. \]
\[ m_i \equiv \int_{x_{i-1}}^{x_i} f(x) \, dx, \text{ where } x_{i-1} \leq x \leq x_i. \]
\[ U(f, N, \alpha) = \sum_{i=1}^{n} M_i \Delta x_i. \]
\[ L(f, N, \alpha) = \sum_{i=1}^{n} m_i \Delta x_i. \]
\[ U(f, N) = \sup_{\alpha} U(f, N, \alpha). \]
\[ L(f, N) = \inf_{\alpha} L(f, N, \alpha). \]

We assume \( f \) is bounded on \([a, b]\). Then there exists an \( M \) and \( m \) such that \( m \leq f(x) \leq M \). If \( \int_{a}^{b} f(x) \, dx \) exists, then \( m(b-a) \leq \int_{a}^{b} f(x) \, dx \leq M(b-a) \). Moreover \( \sum_{i=1}^{n} m_i \Delta x_i = m(b-a) \leq L(f, N) \leq U(f, N) \)

\[ \leq \sum_{i=1}^{n} M_i \Delta x_i = M(b-a). \]

From these inequalities, for a bounded function \( f \), \( \int_{a}^{b} f(x) \, dx \) and \( \int_{a}^{b} f(x) \, dx \) always exists.

Consider the diagram, where

\[ \int_{a}^{b} f(x) \, dx = A_c. \]

\[ A_m \leq A_c \leq A_m \text{, and} \]
\[ m(b-a) \leq A_c \leq M(b-a). \]

Therefore

\[ m(b-a) \leq \int_{a}^{b} f(x) \, dx \leq M(b-a). \]

**Figure 3**

**Definition.** \( N^* \) is a refinement of the net \( N \) if and only if \( N^* \supset N \)

Suppose \( N_1 \) and \( N_2 \) are two nets for \([a, b]\). Then \( N^* \) is the common refinement of \( N_1 \) and \( N_2 \) if and only if \( N^* = N_1 \cup N_2 \).
Theorem XII. If $N^*$ is a refinement of $N$ then $L(f, N^*; \alpha) \leq U(f, N^*; \alpha) \leq U(f, N; \alpha)$.

Proof: Suppose we consider the inequality $L(f, N^*; \alpha) \leq L(f, N^*; \alpha)$, and suppose $N^*$ contains just one point more than $N$, say $x \in \{x_i, x_{i+1}\}$ for some $i$ where $x_{i+1}$ and $x_i$ are two successive points of $N$.

Let $\omega_i = \frac{1}{2} \int f(x) \, dx$ where $x_i \leq x \leq x_{i+1}$ and $\omega_x = \frac{1}{2} \int f(x) \, dx$ where $x = x_i$.

From this we have $m_i \leq \omega_x \leq \omega_1$. Hence $(\omega_1 - m_i) > 0$ and $(\omega_1 - m_i) > 0$.

Observe that $L(f, N^*; \alpha) - L(f, N; \alpha) = U(f, N^*; \alpha) + \omega_2 (x_{i+1} - x_i) - m_i (x_i - x_{i-1}) - (u_1 - m_i) (x_{i-1} - x_i) = 0$.

If $N^*$ has $k$ points more than $N$ we repeat the above argument $k$ times. Hence $L(f, N^*; \alpha) \leq U(f, N^*; \alpha)$. In a similar manner we prove $U(f, N^*; \alpha) \leq U(f, N; \alpha)$.

Theorem XIII. \[ \int_a^b f(x) \, dx = \int_a^b f(x) \, dx. \]

Proof: Let $N_1$ and $N_2$ be nets for the interval $[a, b]$.

By Theorem XI, if $N_1$ is a common refinement of $N_1$ and $N_2$ then $L(f, N_1; \alpha) \leq L(f, N_2; \alpha) \leq U(f, N_2; \alpha) \leq U(f, N_1; \alpha)$. Hence $L(f, N_1; \alpha) \leq U(f, N_1; \alpha)$.

Then the $\frac{1}{2}$ of all $N_1$ gives $f, N_1, \alpha) = \int_a^b f(x) \, dx. But \int_a^b f(x) \, dx \leq \int_a^b f(x) \, dx = U(f, N_1; \alpha).

Observe that $\int_a^b f(x) \, dx = \frac{1}{2} \int f(x) \, dx \leq \frac{1}{2} \int f(x) \, dx = U(f, N_1, \alpha)$; for every $N_2$. Hence $\int_a^b f(x) \, dx = \int_a^b f(x) \, dx$.

Definition. $f \in R(\alpha)$ means $f$ is Riemann - Stieltjes integrable on $[a, b]$, i.e. is Riemann integrable with respect to $\alpha$ on $[a, b]$, (where $\alpha$ is monotone increasing on $[a, b]$).

Theorem XIV. $f \in R(\alpha)$ if and only if for $\epsilon > 0$ there exists a net $N$ such that $U(f, N, \alpha) - L(f, N, \alpha) < \epsilon$.

Proof: (1) Assume for any $\epsilon > 0$ there exists a net $N$ such that $(U - L) < \epsilon$. But $U(f, N, \alpha) \leq \int_a^b f(x) \, dx \leq \int_a^b f(x) \, dx \leq U(f, N, \alpha)$. 

implies that $\int_a^b f \, dx - \int_a^b f \, dx = \mathcal{U}(f, N, \alpha) - \mathcal{L}(f, N, \alpha)$. Hence for the $\epsilon$ and $N$ in our assumption we have $\int_a^b f \, dx - \int_a^b f \, dx < \epsilon$.

Therefore since $\epsilon$ is arbitrary, we have $\int_a^b f \, dx = \int_a^b f \, dx$, i.e., $f \in \mathcal{R}(\alpha)$.

Proof: Assume $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and let $\epsilon > 0$ be given.

Then there exists nets $N_1$ and $N_2$ such that $\mathcal{U}(f, N_1, \alpha) - \int_a^b f \, dx < \frac{\epsilon}{2}$ and $\int_a^b f \, dx - \mathcal{L}(f, N_2, \alpha) < \frac{\epsilon}{2}$. But

$$\int_a^b f \, dx = \int_a^b f \, dx - \int_a^b f \, dx < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

since $f$ is Riemann-Stieltjes integrable. Let $N$ be a common refinement of $N_1$ and $N_2$. Then $\mathcal{U}(f, N, \alpha) < \mathcal{U}(f, N_1, \alpha) \leq \mathcal{U}(f, N, \alpha)$ and $\mathcal{L}(f, N_2, \alpha) < \mathcal{L}(f, N, \alpha) < \mathcal{L}(f, N_2, \alpha) + \epsilon$. Therefore

$$\mathcal{U}(f, N, \alpha) - \mathcal{L}(f, N, \alpha) \leq \mathcal{L}(f, N_2, \alpha) + \epsilon < \epsilon.$$ 

Definition.

$$\mu(N) = \max_i \Delta x_i = |N|, \quad \text{where } 1 \leq i \leq N, \quad \text{where } \mu(N)$$

is a mesh] and $\Delta x_i = \alpha(x_i) - \alpha(x_{i-1})$.

Theorem XV. If $f$ is continuous on $[a, b]$ then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

Proof: Let $\epsilon > 0$ be given in an arbitrary manner. Let $\eta$ be such that $|x' - x''| < \eta < \epsilon$. Then $f$ continuous on $[a, b]$ implies $f$ is uniformly continuous on $[a, b]$. Hence, there exists a $\delta(\epsilon)$ such that $|f(x') - f(x'')| < \epsilon$ when $|x' - x''| < \delta$. Let $\mu(N)$ be so small that $|x_i - x_{i+1}| < \delta$ for $i = 1, 2, 3, \ldots, n$. Then observe;

$$\begin{align*}
\mathcal{U}(f, N, \alpha) - \mathcal{L}(f, N, \alpha) &= \sum_{i=1}^{n} (M_i - m_i) \Delta x_i < \eta \sum_{i=1}^{n} \Delta x_i < \eta |N| \leq \eta (\mathcal{L}(\alpha) - \mathcal{L}(\alpha)) < \epsilon,
\end{align*}$$

since $M_i - m_i < \eta$ for all $i$. Therefore $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

The Riemann-Stieltjes integral as a limit of a sum.

Let $f \in \mathcal{R}(\alpha)$. Set $S(f, N, \alpha) = \sum_{i=1}^{n} f(\gamma_i) \Delta x_i$.

Then consider $\lim_{|N| \to 0} S(f, N, \alpha)$. If this limit exists call it $L$. 


We will show later that \( l = \int_{a}^{b} f \, d\alpha \).

Note - \( f \) is assumed to be bounded and \( \alpha \) is assumed to be monotone increasing.

Theorem XVI. If \( f \in R(\alpha) \) and if \( \alpha \) is continuous, then
\[
\lim_{N \to \infty} S(f, N, \alpha) = \int_{a}^{b} f \, d\alpha.
\]
Conversely if \( \lim_{N \to \infty} S(f, N, \alpha) = \int_{a}^{b} f \, d\alpha \) holds, then \( f \in R(\alpha) \).

Proof: (i): Let \( \epsilon > 0 \) be given arbitrarily. There exists \( \delta(\epsilon) \) such that \( U(f, N, \alpha) < \int_{a}^{b} f \, d\alpha + \frac{\epsilon}{4} \). Let \( M = \max |f(\alpha)| \in [a, b] \).

Take \( \delta_{1} = \frac{\epsilon}{4M} \) where \( n \) is the number of intervals into which \([a, b]\) is divided by \( N \). Let \( N \) be any net for which \( |N| < \delta \).

Consider the sum \( S(f, N, \alpha) \). Each interval of \( n \) which contains a point of \( N \) contributes no more than \( \delta \cdot \mu(N) \cdot (a-v) \in \frac{M}{4Mn} < \frac{\epsilon}{4} \) to \( U(f, N, \alpha) \). Thus \( S(f, N, \alpha) \leq U(f, N, \alpha) + \frac{\epsilon}{4} < \int_{a}^{b} f \, d\alpha + \frac{\epsilon}{4} \)

for \( |N| < \delta_{1} \). In a similar manner we can find \( \delta \) such that if \( |N| < \delta_{2} \) then \( S(f, N, \alpha) > \int_{a}^{b} f \, d\alpha \). Let \( \delta = \min(\delta_{1}, \delta_{2}) \).

Then \( S(f, N, \alpha) < \int_{a}^{b} f \, d\alpha + \frac{\epsilon}{4} \) for \( |N| < \delta \) and \( \delta \) for \( |N| < \delta \).

Hence \( |S(f, N, \alpha) - \int_{a}^{b} f \, d\alpha| < \epsilon \) for all \( |N| < \delta \), i.e., \( \lim_{|N| \to 0} S(f, N, \alpha) = \int_{a}^{b} f \, d\alpha \), a.e., a.e.

Proof: (ii): Assume \( \lim_{|N| \to 0} S(f, N, \alpha) = \int_{a}^{b} f \, d\alpha \).

Then for any \( \epsilon > 0 \) there exists \( \delta(\epsilon) \) such that \( L - \frac{\epsilon}{2} \leq S(f, N, \alpha) \leq L + \frac{\epsilon}{2} \)

for \( |N| < \delta \). Let \( \gamma \in [x_{n-1}, x_{n}] \). Then for a fixed \( n \) taking \( \frac{b}{2} \) and \( \frac{a}{2} \)

and \( \epsilon(\gamma) \in [a, b] \), \( L - \frac{\epsilon}{2} \leq L(f, N, \alpha) \leq U(f, N, \alpha) \leq L + \frac{\epsilon}{2} \) implies \( \lim_{|N| \to 0} |S(f, N, \alpha) - L(f, N, \alpha)| < \epsilon \), whence \( |N| < \delta \).

Properties of the Riemann-Stieltjes Integral.
1. If \( f \) and \( g \in R(\alpha) \) on \([a,b]\) and \( g(x) \geq f(x) \), then
\[
\int_a^b f \, d\alpha \leq \int_a^b g \, d\alpha.
\]

2. If \( f \in R(\alpha) \) on \([a,b]\) and \( c \in [a,b] \), then \( f \in R(\alpha) \) on \([a,c]\) and on \([c,b]\) and
\[
\int_a^b f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha.
\]

3. If \( f \in R(\alpha) \) on \([a,b]\) and if \( f \in R(\beta) \) on \([c,b]\), then
\[
f \in R(\alpha + \beta) \text{ on } [a,b] \text{ and } \int_a^b f \, d(\alpha + \beta) = \int_a^c f \, d\alpha + \int_c^b f \, d\beta.
\]

4. If \( f \in R(\alpha) \) on \([a,b]\) and \( c \) is any non-zero constant,
then \( f \in R(c\alpha) \) and \( \int_a^b f \, d(c\alpha) = c \int_a^b f \, d\alpha \).

5. If \( f \in R(\alpha) \) on \([a,b]\), then \( \int f \, d R(\alpha) \) on \([a,b]\) and
\[
|\int_a^b f \, d\alpha| \leq \int_a^b |f| \, d\alpha.
\]

Proof: We wish to show that \( f \in R(\alpha) \) on \([a,b]\). Let
\( \epsilon > 0 \) be given arbitrarily. Then there exists an \( \mathcal{N} \) such that
\[
|u(f, N, \alpha) - L(f, N, \alpha)| < \epsilon. \quad \text{Note that, (i) } |f(\alpha) - f(\beta)| = |f(\alpha)| - |f(\beta)|.
\]
Let \( M_\alpha = \sup |f(\alpha)| \) where \( X_{i-1} \leq x \leq X_i \) and \( i = 1, 2, 3 \ldots n \).
Let \( m_\alpha = \inf |f(\alpha)| \) where \( X_{i-1} \leq x \leq X_i \) and \( i = 1, 2, 3 \ldots n \).
Observe that
\[
\Delta u(f, N, \alpha) - L(f, N, \alpha) = \sum_{i=1}^{n} (M_\alpha - m_\alpha) \Delta x_i,
\]
\[
= \sum_{i=1}^{n} |M_\alpha - m_\alpha| \Delta x_i,
\]
Since by (i) \( |u(f, N, \alpha) - L(f, N, \alpha)| < \epsilon \)
hence \( f \in R(\alpha) \), Q.E.D.

Proof (ii). If \( f^* = \frac{|f| + f}{2} \) and \( f^- = \frac{|f| - f}{2} \) where \( f^* = \max (0, f) \)
and \( f^- = \min (0, f) \in [a,b] \), then \( |f| = f^* - f^- \) and \( f = f^* - f^- \).

Observe that \( f^* \) and \( f^- \geq 0 \) on \([a,b]\). Now observe that
\[
\left| \int_a^b f \, d\alpha \right| = \left| \int_a^b f_+ \, d\alpha - \int_a^b f_- \, d\alpha \right| \leq \int_a^b f^* \, d\alpha + \int_a^b f^- \, d\alpha.
\]
Therefore \[ \left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha. \]

Theorem XVII. If \( f \) and \( g \in \mathbb{R}(\alpha) \) on \([a,b]\), then \( f \cdot g \in \mathbb{R}(\alpha) \) on \([a,b]\).

Proof: It is sufficient to show that \( f \in \mathbb{R}(\alpha) \) on \([a,b] \)
implies \( f^2 \in \mathbb{R}(\alpha) \) on \([a,b] \).

Let \( M_i = \max \{|f(\omega)| \}, \) and \( m_i = \min \{|f(\omega)| \}, \)
Let \( \varepsilon > 0 \) be given arbitrarily and let \( M \) be such that
\[ |f(\omega)| \leq M \quad \text{on } [a,b]. \]
Then there exists an \( N \) such that
\[ u(1f,N,\alpha) - L(1f,N,\alpha) < \varepsilon. \]
Observe that;
\[ u(1f^2,N,\alpha) - L(1f^2,N,\alpha) \leq 2M \sum_{k=1}^n (M_i - m_i) \Delta \alpha_i = 2M \left[ u(1f,N,\alpha) - L(1f,N,\alpha) \right] \leq 2M \varepsilon, \]
hence \( f^2 \in \mathbb{R}(\alpha) \) on \([a,b], \) a.e.p.

Scope of Stieltjes' Integration.

Definition. Unit step function.
\[ \mathbb{I}(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases} \]

Theorem XVIII. If \( f \) is continuous on \([a,b] \) and if for \( c \in [a,b], \)
\[ \alpha(x) = \mathbb{I}(x-c), \quad \text{then } \int_a^b f(x) \, d\alpha = f(c). \]

Proof: Let \( N \) be any net with \( c \) as a point of subdivision,
then observe that;
\[ S(f,N,\alpha) = \sum_{i=1}^{n} f(x_i) \Delta \alpha_i = f(c) \]
\[ = \sum_{i=1}^{n} f(x_i) \left[ \alpha(x_i) - \alpha(x_{i-1}) \right] = f(c) [1-\alpha] = f(c). \]
Note - If \( C = \emptyset \) the theorem does not hold.

If \( \alpha(x) \) is constant, then \( \int_{a}^{b} f \, d\alpha = 0 \) since \( \Delta \alpha = \Delta x = 0 \).

Theorem XIX. If \( C_n \geq 0 \) and \( \sum_{n=1}^{\infty} C_n \) converges let \( \{x_n\} \) be any sequence of distinct points in \([a, b]\) and let \( \alpha(x) \)

\[
\alpha(x) = \sum_{n=1}^{\infty} C_n \mathbb{1}(x-x_n).
\]

Then \( f \) is continuous on \([a, b]\), \( \int_{a}^{b} f \, d\alpha = \sum C_n f(x_n) \).

Proof: Observe that since \( 0 \leq \mathbb{1}(x-x_n) \leq 1 \), the series \( \alpha(x) = \sum_{n=1}^{\infty} C_n \mathbb{1}(x-x_n) \) converges by the comparison test. i.e. \( C_n \mathbb{1}(x-x_n) \leq C_n \) for all \( n \). The function \( \alpha \) is evidently monotonic, therefore let \( \epsilon > 0 \) be given arbitrarily. Then there exists an \( \eta = \eta(\epsilon) \) such that \( \sum_{n=\eta}^{\infty} C_n < \epsilon \).

Note - \( \alpha(a) = 0 \) and \( \alpha(b) = \sum_{n=1}^{\infty} C_n \).

Set \( \alpha(x) = \sum_{n=0}^{\infty} C_n \mathbb{1}(x-x_n) + \sum_{n=1}^{\infty} C_n \mathbb{1}(x-x_n) \).

\[
\alpha_1(x) = \sum_{n=0}^{\infty} C_n \mathbb{1}(x-x_n).
\]

\[
\alpha_2(x) = \sum_{n=1}^{\infty} C_n \mathbb{1}(x-x_n).
\]

Then \( \int_{a}^{b} f \, d\alpha = \int_{a}^{b} f \, d\alpha_1 + \int_{a}^{b} f \, d\alpha_2 \).

Since \( f \) is continuous on \([a, b]\) there exists \( M > 0 \) such that \( |f(x)| \leq M \) on \([a, b]\). Now observe that \( \int_{a}^{b} f \, d\alpha_1 \)

\[
= \sum C_n \int_{a}^{b} f \, d\mathbb{1}(x-x_n). \quad \text{But} \quad \sum C_n \int f \, d\mathbb{1}_n = \sum C_n f(x_n).
\]

Hence \( \int_{a}^{b} f \, d\alpha - \sum_{n=\infty}^{\eta} C_n f(x_n) = \left| \int_{a}^{b} f \, d\alpha_2 \right| \leq M [\alpha_2(b) - \alpha_2(a)] \).
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Theorem XX. If \( f \in \mathcal{R}(a) \) on \([a,b]\) and if \( \alpha(x) \) is differentiable on \([a,b]\) then
\[
\int_a^b f(x) \, dx = \int_a^b f(x) \alpha'(x) \, dx.
\]

Proof: Let \( N \) be a net for \([a,b]\). Then let \( y_i \in [x_{i-1}, x_i] \) be chosen so that \( \alpha(x_i) - \alpha(x_{i-1}) = \alpha'(y_i) (x_i - x_{i-1}) \).

(By virtue of the Mean Value Theorem.) Now, observe that
\[
\sum_{i=1}^n f(y_i) \Delta x_i = \sum_{i=1}^n f(y_i) \alpha'(y_i) \Delta x_i.
\]
Hence if \( N \to x \) we have
\[
\int_a^b f(x) \, dx = \int_a^b f(x) \alpha'(x) \, dx.
\]
Q.E.D.

The Fundamental Theorem of Calculus.

Theorem XXI. Let \( f \) be Riemann integrable on \([a,b]\). Set \( F(x) = \int_a^x f(t) \, dt \). Then \( F(x) \) is continuous on \([a,b]\).

If \( f \) is continuous at \( x_0 \in [a,b] \) then \( F'(x_0) = f(x_0) \).

Theorem XXII. If \( f \in \mathcal{R} \) on \([a,b]\) and if \( F(x) = \int_a^x f(t) \, dt \) \( \text{i.e.} \)

\( F(x) \) is a primitive of \( f(x) \) then
\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

Both of the above theorems are considered to be fundamental

Theorem XXI. Let \( x_0 \) and \( x_0 + \Delta x \in [a,b] \), where \( \Delta x \neq 0 \).
Then, by Theorem VI, \( F(x_0 + \Delta x) - F(x_0) = \int_{x_0}^{x_0 + \Delta x} f(t) \, dt \)
\[- \int_{x_0}^{x_0 + \Delta x} f(t) \, dt = \int_{x_0}^{x_0 + \Delta x} f(t) \, dt.\]

By Theorem \( \Xi \),
\[
\frac{F(x_0 + \Delta x) - F(x_0)}{\Delta x} = \frac{1}{\Delta x} \int_{x_0}^{x_0 + \Delta x} f(t) \, dt = f(x_0)
\]
for some number \( \xi \), where \( x_0 < \xi < x_0 + \Delta x \). Therefore if \( \Delta x \to 0 \)
\[
\lim_{\Delta x \to 0} \frac{F(x_0 + \Delta x) - F(x_0)}{\Delta x} = f(x_0),
\]
hence the theorem.

Proof II: By Theorem I and the hypothesis of Theorem II, the
two functions \( \int_{a}^{x} f(t) \, dt \) and \( F(x) \) have the same derivative on
\( [a, b] \). Therefore, (by Theorem II, Consequence of M.V.T.), they
differ by a constant, and (4)
\[
\int_{a}^{x} f(t) \, dt = F(x) - C = F(a),
\]
then, substituting in (1) we get \( \int_{a}^{x} f(t) \, dt = F(x) - F(a) \) which,
for the particular value \( x = b \), is the desired result and hence
the theorem.

Bounded Variation.

Definition. Let \( N \) be any net for \( [a, b] \). Let \( f \) be defined on
\( [a, b] \). Then, \( \sup_{[a, b]} \left| f(x) \right| \) is finite, then \( \int_{a}^{x} f(t) \, dt \)
is said to be of bounded variation over \( [a, b] \).

Examples:

1. Monotonic Function.

Let \( f \) be monotone increasing on \( [a, b] \).

Then \( \sup_{[a, b]} \sum_{i=1}^{\infty} \left| f(x_i) - f(x_{i-1}) \right| = \sup_{[a, b]} \sum_{i=1}^{n} \sum_{j=i}^{n} \left| f(x_i) - f(x_{i-1}) \right|. \)
\[= f(b) - f(a) = \int_a^b f'(t) \, dt \leq \max_{a \leq t \leq b} V(f) \]

is of Bounded Variation on \([a, b]\).

2. Any function \(f\) which has a bounded derivative on \([a, b]\)

is of bounded variation there.

Proof: By assumption there exists a \(M > 0\) such that

\[|f(x)| \leq M \quad \text{on} \quad [a, b]\]

Note that \([a, b] = \sum_{k=1}^{n} \frac{b-a}{n} \int_{x_{k-1}}^{x_k} f'(t) \, dt \leq \sum_{k=1}^{n} \frac{b-a}{n} \int_{x_{k-1}}^{x_k} |f'(t)| \, dt \]

\[= \sum_{k=1}^{n} \frac{b-a}{n} \int_{x_{k-1}}^{x_k} |f'(t)| \leq M \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} 1 = M(b-a) < +\infty.

3. A function may be continuous on \([a, b]\) and any net not

be of Bounded variation there. Consider \(f\), defined by;

\[f(x) = \begin{cases} x \sin \frac{1}{x}, & 0 \leq x \leq 2 \\ 0, & x \leq 0 \end{cases}
\]

Then \(V(f) = \lim_{n \to \infty} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \).

Now, \((2 + \frac{1}{2}) + (\frac{2}{3} + \frac{1}{4}) + \cdots + \left(\frac{2}{3n-5} + \frac{2}{3n-6}\right) + \frac{2}{3n-1} > \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{n} \to +\infty \quad \text{as} \quad n \to +\infty \quad \text{since} \quad \sum \frac{1}{n} = +\infty.

Hence \(f\) cannot be of bounded variation on \([a, b]\).

Theorem XXIII: A function of bounded variation on \([a, b]\) is bounded.

Proof: Let \(V(f) = M\), and let \(N\) be a net for \([a, x]\) where \(x < b\). Then observe that \(|f(x) - f(\alpha)| = \sum_{x_k} \int_{x_k}^{x_{k+1}} |f'(t)| \, dt \leq \sum_{x_k} |f(x_k) - f(x_{k-1})| \leq V(f) \).

Hence \(|f(x) - f(\alpha)| \leq M\) on \([a, x]\) for all \(x \in [a, b]\).

\(\text{Therefore } f\text{ is bounded on } [a, b].\)

Theorem XXIV: If \(f\) and \(g\) are of bounded variation on \([a, b]\) then

\[|f(x) - f(\alpha) + g(x) - g(\beta)| \leq |f(x) - f(\alpha)| + |g(x) - g(\beta)| \leq M + M \]
\( f + g \) and \( f \cdot g \) are of bounded variation on \([a,b]\)

Proof (i). Let \( N \) be any net for \([a,b]\). Then
\[
\sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| \leq \sum_{i=1}^{n} \left| g(x_i) - g(x_{i-1}) \right|
\]
Therefore \( \mathcal{V}(f) + \mathcal{V}(g) \). Hence \( f + g \) is of bounded variation on \([a,b]\).

(ii) Let \( \alpha \) and \( \beta \) be positive constants such that
\[
|f(x)| \leq \alpha + |g(x)| \leq \beta \text{ on } [a,b].
\]
Observe that
\[
\sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| \leq \sum_{i=1}^{n} \left| g(x_i) - g(x_{i-1}) \right|
\]
Therefore \( f \cdot g \) is of bounded variation on \([a,b]\).

Corollary.

If \( f \) and \( g \) are monotonically increasing on \([a,b]\), then \( f \cdot g \) is of bounded variation on \([a,b]\).

Definition.

Let \( f \) be of bounded variation on \([a,b]\). Then, for \( x \in [a,b] \), let \( N \) be an arbitrary net for \([a,x]\). Set
\[
\Delta f = f(x) - f(x_{i-1}), \quad \text{and} \quad x = x_0 < x_1 < \cdots < x_n = x-b.
\]
Split the set of integers into two classes \( A \) and \( B \) accordingly as \( \Delta f \) is positive or negative respectively. Then we define
\[
\mathcal{P}(f) = \frac{b-a}{n} \sum_{i=1}^{n} |\Delta f|, \quad \mathcal{N}(f) = \frac{b-a}{n} \sum_{i=1}^{n} |\Delta f|.
\]
\( \mathcal{V}(f) = \frac{b-a}{n} \sum_{i=1}^{n} |\Delta f| \) are called the positive, negative and
total variation function of \( f \) on \([a, b]\) respectively. Observe that
\[
\mathcal{P}(a) = \mathcal{N}(a) = V(a) = 0.
\]
Also, note that \( \mathcal{P}(\infty) \leq V(\infty) \) implies \( N(\infty) \) and \( \mathcal{P}(\infty) \) are finite in \([a, b]\) since \( V(\infty) = V(b) = \frac{\mathcal{V}(f)}{b-a} \).

Theorem XXV

If \( f \) is of bounded variation on \([a, b]\) then there exist monotone increasing functions \( \mathcal{P}, \mathcal{N} \) and \( V \) such that

1. \( f(\infty) - f(a) = \mathcal{P}(\infty) - \mathcal{N}(x) \).
2. \( V(x) = P(\infty) + N(x) \).

Proof: (1) Let \( \mathcal{N} \) be an arbitrary net for \([a, b]\). Then observe that

\[
f(\infty) - f(a) = \sum_{k=1}^{n} [\mathcal{F}(\infty) - \mathcal{F}(\infty)] = \sum_{k=1}^{n} \mathcal{F}(\infty) - \mathcal{F}(\infty) \]

so that \( f(\infty) - f(a) + \sum_{k=1}^{n} \mathcal{F}(\infty) \Rightarrow \sum_{k=1}^{n} \mathcal{F}(\infty) \) implies \( H_{*} + f(\infty) = \mathcal{P}(\infty) + f(a) \)

which in turn implies that \( f(\infty) - f(a) = \mathcal{P}(\infty) - \mathcal{N}(x) \).

Call this inequality (2). Then (1) can be stated as \( \sum_{k=1}^{n} \mathcal{F}(\infty) = f(\infty) - \mathcal{F}(\infty) \)

(2) \( f(\infty) = \mathcal{P}(\infty) + \mathcal{N}(x) \).

This implies that \( f(\infty) = \mathcal{P}(\infty) + \mathcal{N}(x) \), which in turn implies that \( f(\infty) + f(a) = \mathcal{P}(\infty) + \mathcal{N}(x) \).

Call this inequality (3).

Therefore by (2) and (3) \( f(\infty) = \mathcal{P}(\infty) \).

(ii) \( V(x) = \mathcal{P}(\infty) + N(x) \).

Now, adding (4) and (5) we have that \( \sum_{k=1}^{n} \mathcal{F}(\infty) = f(\infty) - f(a) + \sum_{k=1}^{n} \mathcal{F}(\infty) \), which implies \( \sum_{k=1}^{n} \mathcal{F}(\infty) = f(\infty) - f(a) \).
\[ \leq f(x) - f(a) + \nu(x), \] which in turn implies that \( 2 \nu(x) \leq f(x) - f(a) + \nu(x). \) By part one of the theorem \( f(x) - f(a) = \nu(x) - N(x). \) Hence \( 2 \nu(x) = \nu(x) - N(x). \) Therefore \( \nu(x) + N(x) = \nu(x). \)

Corollary.

If \( f(x) \) is of bounded variation on \([a,b]\) then \( f(x^+) \) exists for any \( x \in [a,b] \) and \( f(x^-) \) exists for any \( x \in (a,b). \) Moreover, the set of points \( x \in [a,b] \) where \( f \) is discontinuous is at most denumerable.

Theorem XXIV.

If \( f \) is of bounded variation on \([a,b]\) and if there exist functions \( P \) and \( N \) on \([a,b]\) such that \( f(x) = P(x) - N(x), \) we have that \( \nu(x) = P(x) - N(x) \) and \( \nu(x) = P(x) - N(x). \) where \( P \) and \( N \) are positive and negative variation functions of \( f \) on \([a,b]\) and \([a,b] \)

Proof: Since the addition of a constant to a function does not change its total variation, we may assume that \( P(a) = N(a) = 0. \) Then \( \nu(x) = P(x) - N(x). \) Hence \( \nu(x) = P(x) - N(x). \) Therefore \( P(x) + N(x) = \nu(x) \) and \( \nu(x) = P(x) + N(x). \) Adding \( (ii) \) and \( (iii) \) we have that \( P(x) + N(x) = \nu(x) \) and \( \nu(x) = P(x) + N(x). \) Subtracting \( (ii) \) and \( (iii) \) we have that \( P(x) = \nu(x). \)
\[ N(b) \leq N(a), \text{ i.e., } V(N) \leq V(N') \]

Generalization of the Riemann – Stieltjes Integral.

Definition.

Let \( f \) and \( g \) be defined on \([a, b]\). Let \( N \) be any net for \([a, b]\). Then \( f \) is said to be integrable with respect to \( g \) over \([a, b]\) if and only if \( \lim_{|N| \to 0} \sum (f, N, g) \) exists and is finite. If this limit exists and is finite, then we say \( f \in R^*(g) \) and

\[
\lim_{|N| \to 0} \sum (f, N, g) = \int_a^b f \, dg.
\]

Remarks. If \( f \) is monotone increasing then \( R^*(g) = R(g) \).
If \( g \) is differentiable on \([a, b]\) and \( g' \) is continuous on \([a, b]\) then

\[
\int_a^b f \, dg = \int_a^b f \, \mathrm{g}'.
\]

Properties of the Riemann – Stieltjes Integral.

(a) If \( f \in R^*(g) \) and \( f_2 \in R^*(g) \) on \([a, b]\), then \( f_1 + f_2 \in R^*(g) \) on \([a, b]\) and

\[
\int_a^b (f_1 + f_2) \, dg = \int_a^b f_1 \, dg + \int_a^b f_2 \, dg.
\]

(b) If \( f \in R^*(g_1) \) and \( f \in R^*(g_2) \) on \([a, b]\), then \( f \in R^*(g_1 + g_2) \) on \([a, b]\) and

\[
\int_a^b f \, (g_1 + g_2) = \int_a^b f \, g_1 + \int_a^b f \, g_2.
\]

(c) \( \int_a^b f \, dg = -\int_a^b f \, dg. \)

(d) If \( f \in R^*(g) \), then \( cf \in R^*(g) \) and \( f \in R^*(cg) \) and for any constant \( c \) then

\[
\int_a^b cf \, dg = \int_a^b f \, (cg) = c \int_a^b f \, dg.
\]

(e) If \( g \) is constant on \([a, b]\) then every function \( f \in R^*(g) \) and

\[
\int_a^b f \, dg = 0.
\]
(f) If \( a < c < b \) and \( f \in K^*(g) \) on \([a, b]\), then
\[
\int_a^b f \, dg = \int_a^c f \, dg + \int_c^b f \, dg.
\]

All of the above properties follow from the definition of \( f \in K(g) \).

Notations:
1. \( f \in K \) if and only if \( \lim_{|N| \to 0} S(N, f) \) exists and corresponds to \( \int_a^b f \, dx \).

2. \( f \in K^*(g) \) where \( g \) is monotone increasing, if and only if \( \lim_{|N| \to 0} S(N, f, g) \) exists and corresponds to \( \int_a^b f \, dx \).

3. \( f \in K^*(g) \) where \( g \) is not necessarily monotone increasing if and only if \( \lim_{|N| \to 0} S(N, f, g) \) exists and corresponds to \( \int_a^b f \, dx \).

Theorem XXVII.

If \( f \in K^*(g) \) on \([a, b]\), then \( g \in K^*(f) \) on \([a, b]\) and
\[
\int_a^b g \, df = f(b) g(b) - f(a) g(a) - \int_a^b f \, dg.
\]

Proof: Let \( N \) be any net for \([a, b]\). In particular, let \( N \) be the set of points \( N = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\} \) where \( x_0 < x_1 < x_2 < \cdots < x_n \).

Let \( f \in [x_{i-1}, x_i] \), let \( \overline{N} = \{\overline{x_0} = \overline{f_0} < \overline{f_1} < \overline{f_2} < \cdots < \overline{f_{n+1}} = \overline{b}\} \),
where \( \overline{f_0} < \overline{f_1} < \overline{f_2} < \cdots < \overline{f_{n+1}} \). Observe that \( \overline{x}_{i-1} < x_{i-1} < \overline{x}_i \).
where \( i = 1, 2, 3, \ldots, n+1 \). By our partial summation formula above \( S(N, f, g) = \sum_{i=1}^n g(\overline{f_i}) [f(\overline{x}_i) - f(\overline{x}_{i-1})] = f(b) g(b) - f(a) g(a) - \sum_{i=1}^n f(x_{i-1}) [g(\overline{f_i}) - g(\overline{f_{i-1}})] = f(b) g(b) - f(a) g(a) - S(f, \overline{N}, \overline{g}). \)
Now letting $|N| \to 0$ implies $N \to \infty$ and we have that
$$\lim_{|N| \to 0} (N, \delta_f) \cdot \mathcal{F} = \left( \int_{\omega} g(\omega) - \int_{\omega} g(a) - \lim_{|N| \to 0} \delta(N, \delta_f) \right).$$

Hence the theorem.

Corollary 1.

If $f'$ and $g'$ are continuous on $[a, b]$ then
$$\int_a^b g' d\lambda = \int_a^b g f' d\lambda \quad \text{and} \quad \int_a^b g' d\lambda = \int_a^b f g' d\lambda \quad \text{and} \quad \int_a^b g d\lambda = f(b) g(b) - f(a) g(a) - \int_a^b g' d\lambda.$$

Corollary 2.

If $f$ is of bounded variation on $[a, b]$ and if $g$ is continuous on $[a, b]$ then $f \in K^R(g)$ on $[a, b]$.

Proof: It is sufficient to show that $g \in K^R(f)$.

Since $f$ is of bounded variation then $f(x) - f(a) = P(\omega - N(\omega))$.

Let $f = c f (f - N)$ then $\int_a^b g d\lambda = \int_a^b g d\mu + \int_a^b g d\lambda$.

Hence the theorem.

Theorem XXVIII.

If $f$ and $\phi$ are continuous on $[a, b]$ and if $\phi$ is strictly increasing on $[a, b]$ and $f$ is the inverse function of $\phi$ then
$$\int_a^b f d\lambda = \int_{\phi(a)}^{\phi(b)} f (\phi(\lambda)) d\lambda , \text{ where we assume } \gamma = \phi(x).$$

Proof: Let $N$ be the net where $a = \gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n = b$.

Note that $\gamma_i = \phi(\lambda)$ where $\lambda = 0, 1, 2, 3, \cdots, n$. Now consider the net $\overline{N} = \{ \phi(0) = \gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_n = \phi(b) \}$, where $\gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n$. Then observe that for $\overline{\xi} \in [X, X']$

where $a = 1, 2, 3, \cdots, n$,
$$\sum_{i=1}^{n} f(\overline{\xi}_i) (X_i - X_{i-1}) = \sum_{i=1}^{n} \left[ \psi(\gamma_i) \right] \left[ \psi(\gamma_i) - \psi(\gamma_{i-1}) \right].$$
Now, letting $|N| 	o 0$ we have that $|\tilde{u}| \to 0$ by uniform continuity of $\phi$. We have also that $\lim_{|N| \to 0} \frac{1}{n} \sum_{i=1}^{n} f \left( \frac{\chi_i - \chi_{i-1}}{\chi_i - \chi_{i-1}} \right) = \begin{bmatrix} 0 \end{bmatrix}$.

$\lim_{|N| \to 0} \sum_{i=1}^{n} f(\psi(y_i)) \left[ \psi(y_i) - \psi(y_{i-1}) \right]$ implies $\int_{a}^{b} f \, dx = \int_{a}^{b} f(\psi(y)) \, dy$.

**Theorem XXXV. First Mean Value Theorem.**

If $f$ is continuous and $\alpha$ is monotonic on $[a, b]$ then there exists a point $x \in [a, b]$ such that $\int_{a}^{b} f \, dx = f(\alpha) \left[ \alpha(b) - \alpha(a) \right]$.

Proof: Let $M$ be the $|u_b|$ of $f(t)$ where $t \in [a, b]$, and $m = \frac{1}{b-a} \int_{a}^{b} f \, dx$. Then $M \left[ \alpha(b) - \alpha(a) \right] \leq \int_{a}^{b} f \, dx$.

But since $f$ is continuous on $[a, b]$ there exists a point $x \in [a, b]$ such that $\lambda = f(\alpha)$. Therefore $\int_{a}^{b} f \, dx = f(\lambda) \left[ \alpha(b) - \alpha(a) \right]$.

**Remarks.** It may not be possible to select a point $x \in (a, b)$, for consider $\alpha(x) = \{ 0, 1 \}$. Observe that

$\exists = x_0 < x_1 < \cdots < x_n$ and $\sum_{i=1}^{n} f(x_i - x_{i-1}) \left[ \alpha(x_i) - \alpha(x_{i-1}) \right] = f(\alpha)$.

Hence $\int_{a}^{b} f \, dx = f(\alpha) \left[ \alpha(b) - \alpha(a) \right]$.

**Theorem XXXVI. Second Mean Value Theorem.**

If $f$ is monotonic and $\phi$ is continuous, then there exists a point $x \in [a, b]$ such that $\int_{a}^{b} f \, dx = f(\alpha) \left[ \phi(x) - \phi(a) \right] + f(\omega) [\phi(b) - \phi(a)]$. By Theorem XXXVII we have $\int_{a}^{b} f \, dx$.
\[ \int_a^b \frac{d}{dx} f(x) \, df = f(b) \frac{d}{dx} f(a) - f(a) \frac{d}{dx} f(b) - \int_a^b \frac{d}{dx} df. \quad \text{Now by the first mean value theorem there exist a point } \lambda \in [a, b] \text{ such that} \\
\int_a^b \frac{d}{dx} f = \frac{d}{dx} f(\lambda) [f(b) - f(a)]. \quad \text{Hence} \\
\int_a^b \frac{d}{dx} f = f(a) \left( \frac{d}{dx} f(x) - \frac{d}{dx} f(a) \right) + f(b) \left( \frac{d}{dx} f(b) - \frac{d}{dx} f(\lambda) \right). \quad \text{Q.E.D.} \]
