8-1-1962

On affine transformations

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ON AFFINE TRANSFORMATIONS

A THESIS
SUBMITTED TO THE FACULTY OF ATLANTA UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF MASTER OF SCIENCE

BY
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DEPARTMENT OF MATHEMATICS

ATLANTA, GEORGIA
AUGUST 1962
ACKNOWLEDGEMENTS

The writer wishes to express her sincere appreciation to Dr. Subhash C. Saxena for his patience, guidance, and invaluable criticisms which were instrumental in the completion of this thesis.

Appreciation is also expressed to Dr. Lonnie Cross for his vital instruction and inspiration and to The National Science Foundation for its financial assistance.
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CHAPTER I

INTRODUCTION

Nature, Definition and Types of Affine Transformations.--
The group of affine transformations forms a subgroup of the group of projective transformations. Therefore, we can place certain restrictions on the latter group and thereby obtain transformations which leave the fixed ideal line invariant. The restricted group is referred to as the affine group. In order to fully understand the nature of affine transformations we must understand their relationship to projective transformations.

A projective transformation carries an ideal point to an ordinary point or to an ideal point, and an ordinary point to an ordinary point or an ideal point. The concept of ideal point will be discussed later in this chapter. Properties which remain invariant under projective transformations are collinearity, concurrency, cross ratio, and harmonic set.

Given a projective transformation in a plane which has the form

\[
\begin{align*}
    x_1' &= a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \\
    x_2' &= a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \\
    x_3' &= a_{31} x_1 + a_{32} x_2 + a_{33} x_3,
\end{align*}
\]

where the determinant of the matrix of the coefficients

\[
\begin{vmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{vmatrix} = 1
\]
is different from zero.

The projective transformation becomes an affine transformation if an ordinary point goes over to an ordinary point and an ideal point goes over to an ideal point, where \( x_3 = 0 \) always implies \( x'_3 = 0 \). If \( x_3 = 0 \), then

\[
x'_3 = a_{31} x_1 + a_{32} x_2 = 0,
\]

i.e., \( a_{31} x_1 + a_{32} x_2 = 0 \) for all values of \( x_1 \) and \( x_2 \). This necessarily means that \( a_{31} = 0 \) and \( a_{32} = 0 \) but \( a_{33} \neq 0 \), for if it were, then the determinant of the coefficients would be zero. It follows that the general affine transformation in a plane is given by

\[
(1-2) \quad \begin{align*}
x'_1 &= a_{11} x_1 + a_{12} x_2 + a_{13} x_3, \\
x'_2 &= a_{21} x_1 + a_{22} x_2 + a_{23} x_3, \\
x'_3 &= a_{31} x_1 + a_{32} x_2 + a_{33} x_3.
\end{align*}
\]

There is no loss of generality if we assume that this transformation can be written as

\[
(1-3) \quad \begin{align*}
x'_1 &= a_{11} x_1 + a_{12} x_2 + a_{13} x_3, \\
x'_2 &= a_{21} x_1 + a_{22} x_2 + a_{23} x_3, \\
x'_3 &= a_{33} x_3.
\end{align*}
\]

where the determinant of the coefficients

\[
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
0 & 0 & 1
\end{vmatrix}
\]

is different from zero. As nonhomogeneous coordinates we have
It follows clearly from the representation of an affine transformation of a plane onto itself that an affine transformation of a line onto itself may be obtained from the projective transformation on a line

\[(1-5)\]
\[
\begin{align*}
    x'_1 &= a_{11} x_1 + a_{12} x_2, \\
    x'_2 &= a_{21} x_1 + a_{22} x_2 \quad (a_{11} a_{22} - a_{12} a_{21} \neq 0),
\end{align*}
\]

if and only if \( x_2 = 0 \) implies \( x'_2 = 0 \), that is to say, if and only if \( a_{22} = 0 \). If \( a_{11} a_{22} - a_{12} a_{21} = 0 \), then \( a_{11} \neq 0 \).

Since, for any real number \( k \), \((x_1, x_2) = (kx_1, kx_2)\), we can represent the homogeneous coordinates of any ordinary point by the set of its homogeneous coordinates for which \( a_{22} = 1 \). The resulting affine transformation may be expressed in the form

\[(1-6)\]
\[
\begin{align*}
    x'_1 &= a_{11} x_1 + a_{12} x_2, \\
    x'_2 &= a_{22} x_2,
\end{align*}
\]

where the corresponding determinant of the coefficients

\[
\begin{vmatrix}
    a_{11} & a_{12} \\
    0 & 1
\end{vmatrix}
\]

is different from zero, i.e., \( a_{11} \neq 0 \). As homogeneous coordinates of a point on a line we have

\[
x' = a_{11} x + a_{12}, \quad a_{11} \neq 0.
\]

By the same reasoning, an affine transformation on a three-space may be expressed in the form
\[
x_1' = a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + a_{14} x_4,
\]
\[
x_2' = a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + a_{24} x_4,
\]
\[
x_3' = a_{31} x_1 + a_{32} x_2 + a_{33} x_3 + a_{34} x_4,
\]
\[
x_4' = x_4,
\]
where the corresponding determinant of coefficients is different from zero.

There are two observations that should be made regarding the representations of affine transformations. First, we note that for affine transformations the representation in terms of homogeneous coordinates is equivalent to the representation in terms of nonhomogeneous coordinates. The second observation is concerned with the identity matrices. Any affine transformation may be expressed such that the last row of its square matrix is identical with the last row of the corresponding identity matrix. The last rows of the square matrices provide convenient means of recognizing matrices of affine transformations. A square matrix with determinant different from zero may be used to represent an affine transformation if and only if its last row may be obtained from the last row of the corresponding identity matrix by multiplying each element by a non-zero constant. \[6;164-65\].

The general affine transformation has been discussed above and, in general, carries parallel lines into parallel lines, finite points into finite points, straight lines into straight lines, and tangent curves into tangent curves. It leaves the ideal line fixed and it may not preserve
angles areas. We may define the affine transformation as being any projective transformation under which ordinary points correspond to ordinary points and ideal points correspond to ideal points.

In our study of affine transformations, we find that some have particular characteristics which cause them to differ slightly from some others. The homogeneous affine transformation is one in which each constant term is zero. Its form is

\[(1-8) \quad x' = a_{11} x + a_{12} y, \]
\[y' = a_{21} x + a_{22} y,\]

where \(a_{11}a_{22} - a_{12}a_{21} \neq 0\). The isogonal affine transformation is one which does not change the size of angles. It has the form

\[(1-9) \quad x' = a_{11} x + a_{12} y + a_{13}, \]
\[y' = a_{21} x + a_{22} y + a_{23}, \]

where either \(a_{11} = a_{22}\) and \(a_{21} = -a_{12}\) or \(-a_{11} = a_{21}\) and \(-a_{12} = a_{22}\). An affine transformation is singular if \(a_{11}a_{22} - a_{12}a_{21} = 0\), and an affine transformation is nonsingular if \(a_{11}a_{22} - a_{12}a_{21} \neq 0\).

**Ideal Points.**—Since points are inevitably associated with lines and planes, in our discussion of ideal points we designate an arbitrary plane in projective space as the ideal plane or plane at infinity. Any two planes in projective geometry are equivalent. Therefore, we may use the projective transformation of a space onto itself so that the ideal plane has the equation \(x = 0\). Then the points \((x, x, x, 0)\)
are on the ideal plane and are called ideal points or points at infinity. All other points or the ordinary points have homogeneous coordinates of the form \((x, x_2, x_3, 1)\) and may be represented by nonhomogeneous coordinates \((x, y, z)\). Similarly, the lines on the ideal plane are called ideal lines. All other lines are called ordinary lines. All planes that are distinct from the ideal plane are called ordinary planes. Every ordinary plane contains exactly one ideal line since two distinct planes in projective three-space have exactly one line in common. Likewise every ordinary line contains exactly one ideal point.

We shall assume that the coordinate system has been chosen such that on any line of points \((x, x_3)\) the ideal point is taken as \((1, 0)\), on any plane of points \((x, x_2, x_3)\) the ideal line is taken as the line \(x_3 = 0\) with points \((x, x_2, 0)\), and on the three-space of points \((x, x_2, x_3, x_4)\) the ideal plane is taken as the plane \(x_4 = 0\) with points \((x, x_2, x_3, 0)\). If we delete the ideal plane from projective three-space, the remaining space is called affine three-space. Similarly, a projective plane with the ideal line deleted is called an affine plane; a projective line with its ideal point deleted is called an affine line [6;150-51].

We can denote the line at infinity by \(l_\infty\). However, it should not be thought of as necessarily "at infinity"; only if we desire the resulting geometry to be intuitionally equivalent to the ordinary elementary geometry, i.e., if we desire
the figures of the geometry to "look like" the figures we are familiar with, is it necessary for us to think of $1/\infty$ as being at infinity [10;153].

Parallels— Two ordinary lines on a plane which do not intersect in an ordinary point are said to be parallel, i.e., their intersection is in an ideal point. Similarly, two ordinary planes which intersect in an ideal line are said to be parallel. A line is parallel to a plane if and only if it intersects the plane in an ideal point.

We pointed out previously that the ideal plane $x_4 = 0$ remains fixed under any affine transformation. Every ideal point $(x_1, x_2, x_3, 0)$ must correspond to an ideal point $(x'_1, x'_2, x'_3, 0)$, However, a particular ideal point does not necessarily correspond to itself. Therefore, the affine transformation has the property of carrying parallel lines into parallel lines and parallel planes into parallel planes, i.e., parallelism is an affine invariant.

The above definitions provide the basis for many of the figures and properties that are studied in an elementary geometry course. The most common closed figure based on parallelism is the parallelogram. In euclidean plane geometry, a parallelogram is often defined to be a simple plane quadrilateral with its opposite sides parallel. A simple plane quadrilateral is a figure consisting of four sides and their four successive intersections in pairs as shown in Figure 1.1. As distinguished from the simple plane quadrilateral, we
define a complete quadrilateral to be a figure having four coplanar lines, no three of which are concurrent, and their six points of intersection. If two of the opposite vertices meet in an ideal point, then the opposite sides are parallel. Now we can define a parallelogram as being a complete quadrilateral with two of its opposite vertices on the ideal line. When the ideal line $l_\infty$ of affine geometry is represented by a line of a euclidean plane, a parallelogram abcd may appear as in either of the two figures below.

**Mid-point.** Let AB be a line segment of a line and let $P_\infty$ be the ideal point of the line. If C is the mid-point of
AB, then four points AB, CP∞ from a harmonic set H(AB, CP∞), where a harmonic set is defined to be the set of four collinear points whose cross ratio is -1, i.e., four points A, B, C, and D form a harmonic set if and only if the cross ratio of these four points is -1. If (AB, CD) form a harmonic set, then C and D divides AB internally and externally in the same ratio and AC/BC = AD/BD. In other words, the mid-point of the segment AB is the harmonic conjugate with respect to A and B of the ideal point P∞ on the line AB and the mid-point of AB is C, where we have H(AB, CP∞). Analytically, the mid-point of a segment AB, where A:(x, y, z) and B: (x', y', z') is defined to be \((x/2+x/2, y/2+y/2, z/2+z/2)\).

We now use the above definitions and the invariance of harmonic sets under affine transformations to prove three well known theorems.

**Theorem 1.** The diagonals of a parallelogram bisect each other.

![Figure 1.4](image)
Proof: Let $ABCD$ be a parallelogram on a plane with ideal line $l_{\infty}$, where $AB \cdot CD = A_{\infty}$, $AD \cdot BC = B_{\infty}$, $AC \cdot l_{\infty} = C_{\infty}$, $BD \cdot l_{\infty} = D_{\infty}$, and $AC \cdot BD = E$. It will be sufficient to prove that $E$ is the mid-point of the segment $AC$ and $BD$. Consider the quadrangle $ABCD$ with $H(A_{\infty}B_{\infty}, C_{\infty}D_{\infty})$ and the perspectivities

$$DBED_{\infty} \stackrel{\propto}{\sim} A_{\infty}B_{\infty}C_{\infty}D_{\infty} \stackrel{D}{\sim} CAC_{\infty}E.$$  

Then since harmonic sets correspond to harmonic sets under projective transformations; we have $H(DB_{\infty}, D_{\infty}E)$ and $H(CA_{\infty}, C_{\infty}E)$, which by definition $E$ is the mid-point of $BD$ and of $AC$. Thus the proof is complete [6;156-57].

Theorem 2. The medians of a triangle are concurrent.

![Figure 1.5](image)

Proof: In order to fully understand this proof it is suggested that we review the proof of Desargues Theorem in
projective geometry. However, the proof will not be considered here. Consider any triangle ABC on a plane with ideal line $l_\infty$. Let $AB \cdot l_\infty = C_\infty$, $BC \cdot l_\infty = A_\infty$, and $AC \cdot l_\infty = B_\infty$. Then $H(C_\infty, B_\infty, A_\infty, D_\infty)$ determines a unique point $D_\infty$ on $l_\infty$. Also there exists a homogeneous coordinate system on the plane with such points as $A:(0, 0, 1)$, $B:(2, 0, 1)$, $C:(0, 2, 1)$, and $D:(1, 1, 0)$. Then the mid-point of BC is $AD \cdot BC = A_\infty:(1, 1, 1)$; the mid-point of AB is $C_\infty:(1, 0, 1)$; and the mid-point of AC is $B_\infty:(0, 1, 1)$. The AA has the equation

$$\begin{vmatrix} x & y & 1 \\ 1 & 1 & 1 \end{vmatrix} = x - y = 0,$$

$BB_\infty$ has equation $x + 2y - 2 = 0$, $CC_\infty$ has equation $2x + y - 2 = 0$. All three equations are satisfied by $(2/3, 2/3, 1)$, and therefore the medians $AA_\infty$, $BB_\infty$, and $CC_\infty$ are concurrent.

Since ABC was an arbitrary triangle, this completes the proof [6;157-58].

Theorem 3. A line bisecting two sides of a triangle is parallel to the third side.

Proof: Let ABC be an arbitrary triangle. Extend lines AC, AB, and BC to intersect $l_\infty$ at $C_\infty$, $A_\infty$, and $B_\infty$ respectively. Let a line bisect AB at a point D, another line bisect AC at a point E, and a third line bisect BC at a point F. Then we have that the point D is the mid-point of AB, where $H(AB, DA_\infty)$, F is the mid-point of BC, where $H(BC, FB_\infty)$
and $E$ is the mid-point of $AC$, where $H(AC, EC\_\infty)$. If the line which bisects $AB$ at $D$ passes through the mid-point $F$ of $BC$, it will intersect the line $AC$ at $C\_\infty$. Similarly, if the line which bisects $AC$ at $E$ passes through the mid-point $F$ of $BC$, it will intersect the line $AB$ at $A\_\infty$. Finally, if the line which bisects $AB$ at $D$ passes through the mid-point $E$ of $AC$, it will intersect $BC$ at $B\_\infty$. Then it follows from the definition of parallel lines that line $EF$ which bisects the sides $AC$ and $BC$ of the given triangle is parallel to the third side of $AB$. Similarly, the line $DE$ which bisects the sides $AB$ and $AC$ of the given triangle is parallel to the third side $BC$. Finally, if the line $DF$ which bisects the sides $AB$ and $BC$ of the given triangle is parallel to the
third side AC. Since ABC is an arbitrary triangle and in the affine plane any triangle is equivalent to any other triangle in the affine plane, the proof is complete.
CHAPTER II

HOMOTHETIC TRANSFORMATIONS

Nature and Definition of the Homothetic Transformation.—
Recall that the group of transformations which leaves the ideal line invariant forms the group of affine transformations. If an affine transformation leaves the ideal line pointwise invariant it is called a homothetic transformation. The set of homothetic transformations forms a group which is a subgroup of the affine transformations and consequently a subgroup of the projective transformations.

We now consider the general affine transformation

\[ x'_1 = a'_{11} x_1 + a'_{12} x_2 + a'_{13} x_3, \]
\[ x'_2 = a'_{21} x_1 + a'_{22} x_2 + a'_{23} x_3, \]
\[ x'_3 = a'_{31} x_1 + a'_{32} x_2 + a'_{33} x_3, \]

and the necessary conditions that it be a homothetic transformation. If any transformation keeps the three points (1, 0, 0), (0, 1, 0), and (1, 1, 0) fixed, then it is a homothetic transformation because if three ideal points remain fixed all ideal points will remain fixed. We can consider the points mentioned because if three points on a line are invariant under a projective transformation then all the points remain invariant. If (1, 0, 0) remains fixed, and since (1, 0, 0) = (k, , 0, 0), then (1, 0, 0) implies (k , 0, 0) under a homothetic transformation. The same reasoning holds for (0, 1, 0) and (1, 1, 0).
First, let us consider the point \((1, 0, 0)\) which implies \((k_1, 0, 0)\) and equating it to the corresponding point under the general affine transformation and obtain \(k_1 = a_{11}, 0 = a_{12}\) and by the same argument \((0, 1, 0)\) implies \((0, k_2, 0)\) and we obtain \(0 = a_{21}, k_2 = a_{22}\). Further we have that \((1, 1, 0)\) implies \((k_3, k_3, 0)\), where \(k_3 = a_{11}\) and \(k_3 = a_{22}\). Therefore, we have \(a_{11} = a_{22} \neq 0\) and \(a_{21} = a_{22} = 0\). From this we can represent any homothetic transformation in the form

\[
\begin{align*}
x_1' &= a_{11}x_1 + a_{13}x_3, \\
x_2' &= a_{21}x_2 + a_{23}x_3, \\
x_3' &= x_3, \quad a \neq 0,
\end{align*}
\]

where the corresponding determinant of the matrix of coefficient takes the form

\[
\begin{vmatrix}
a & a_{13} \\
0 & a_{23} \\
0 & 1
\end{vmatrix} \neq 0,
\]

or as nonhomogeneous coordinates of the points,

\[
x' = ax + a_{13},
\]

\[
y' = ay + a_{23}, \text{ where } a \neq 0.
\]

On a projective plane any projective transformation that leaves all points invariant on an invariant line must also leave all lines invariant on an invariant point (principle of planar duality). Thus every homothetic transformation leaves the points of a pencil of points invariant and the lines of a pencil of lines invariant. The axis of the pencil
of invariant points is called an axis of the homothetic transformation; the center of the pencil of invariant lines is called a center of the homothetic transformation. If a homothetic transformation has two distinct axes of invariant points, it is the identity transformation. If a homothetic transformation has two distinct centers of invariants lines, it is the identity transformation. Since every homothetic transformation has at least one axis and at least one center, every homothetic transformation has a unique axis and a unique center if it is not the identity transformation [6;166].

Homothetic transformations are of different forms depending upon their axis or center. A homothetic transformation is a translation if its center is an ideal point, a dilation if its center is an ordinary point, a point reflection if it is a dilation of period two. We shall discuss each of these in the following paragraphs.

Translations. We now place restrictions on the homothetic transformation and thereby obtain a special type of transformation known as a translation. Clearly, the set of translations on a plane form a subgroup of the group of affine transformations. A homothetic transformation is said to be a translation if it has an ideal point as a center. We can say, then, that a translation simply moves any plane figure a certain distance in a given direction. In other words, if we were to consider a region S, a translation would carry each point of S a given direction by one and the same given
distance into some region $S'$ as is shown in Figure 2.1. Thus when a window is raised, a pane of glass in the window experiences a translation. The region considered is not of particular importance because it is usually desirable to consider the whole unbounded plane. What is important, however, is the law which connects the initial position of an arbitrary point of $S$ with its final position [7;330].

We now want to represent a translation analytically. Let us recall the representation of the homothetic transformation given in (2-1). We want to place certain restrictions on the coefficients and thus obtain the general form of a translation. There is a unique invariant point $(a_{x}, a_{y}, 1 - a_{\nu})$ if and only if $(a_{x}, a_{y}, 1 - a_{\nu}) \neq (0, 0, 0)$. Hence, it is clear that if $a_{\nu} = 1$, there is a unique ideal invariant point if $(a_{x}, a_{y}, 0) \neq (0, 0, 0)$ and all points are invariant. If all points are invariant the general affine transformation becomes an identity. On the ideal line $x_2 = 0$, we have an identity if and only if the points $(1, 0, 0), (0, 1, 0)$, and
(1, 1, 0) are invariant, i.e., if and only if $a_{12} = a_{21} = 0$ and $a_{13} = a_{23}$. (See the discussion on derivation of the general homothetic representation). If $a_{11} \neq 1$, there is a unique ordinary invariant point. Thus the homothetic transformation represents a translation if and only if $a_{11} = 1$. Hence, as homogeneous coordinates we write the translation in the form
\[
(2-2) \quad x'_1 = x_1 + a_3 x_3, \\
x'_2 = x_2 + a_2 x_3, \\
x'_3 = x_3.
\]
where the corresponding matrix takes the form
\[
(2-3) \quad \begin{vmatrix}
1 & 0 & a \\
0 & 1 & b \\
0 & 0 & 1
\end{vmatrix}.
\]

In the nonhomogeneous coordinates system for which $l_\infty$ is the ideal line, it is evident that any translation parallel to the x-axis can be expressed by the equations
\[
x' = x + a, \\
y' = y.
\]
By definition of addition, the first of these equations represents any parabolic projectivity on the x-axis of the kind desired, while the second equation insures the fact that every line parallel to the x-axis is transformed into itself. Similarly, any translation parallel to the y-axis is expressed by the equations
\[
x' = x, \\
y' = y + b.
\]
It is clear that in terms of any nonhomogeneous coordinate system we can express a translation in the form
\begin{equation}
(2-4)
\begin{array}{c}
x' = x + a, \\
y' = y + b.
\end{array}
\end{equation}

**Dilation.**— In terms of the coefficients of (2-1), we obtain a dilation if and only if \( a \neq 1 \). The dilation is, therefore, a homothetic transformation which has a unique ordinary point as its center, where \( a \neq 0, a \neq 1 \). The dilation is not the identity transformation but a projective transformation which leaves every point invariant on the ideal line and leaves every line invariant on a unique ordinary point. Thus any dilation is expressed in the form
\begin{equation}
(2-5)
\begin{array}{c}
x' = ax + c_1, \\
y' = ay + c_2 \quad (a \neq 0, a \neq 1),
\end{array}
\end{equation}

where
\begin{equation}
(2-6)
\begin{vmatrix}
a & 0 & c_1 \\
0 & a & c_2 \\
0 & 0 & 1
\end{vmatrix}
\end{equation}
is the corresponding matrix.

An involution or a transformation of period two is a transformation that is different from the identity but when applied twice obtains the identity, that is to say, it is a transformation whose square is the identity. To illustrate this suppose we consider the transformation
where

\[
\begin{vmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{vmatrix} \neq 0,
\]
i.e., \((a_{11} a_{22} - a_{12} a_{21}) \neq 0\,\), and find the necessary and sufficient condition that this transformation is an involution.

We will call this transformation \(T\). Therefore,

\[
T^2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 + a_{12} a_{21} & a_{11} a_{12} + a_{12} a_{22} \\ a_{21} a_{11} + a_{22} a_{21} & a_{21} a_{12} + a_{22} a_{22} \end{bmatrix}.
\]

In order for this product to be the identity transformation, we must have \((a_{11} a_{22} - a_{12} a_{21}) = 0\) and \((a_{21} a_{11} + a_{22} a_{21}) = 0\). From this relation we see that either

\[
a_{11} + a_{22} = 0\,\text{ or }\, a_{11} = -a_{22}.\]

It is clear that \(a_{12}\) and \(a_{21}\) can not be zero because this would mean that \(T\) is the identity which is contrary to our definition. Hence, we have the necessary condition that the transformation \(T\) is an involution.

To find the sufficient condition we set \(a_{11} = a\,\), \(a_{12} = b\,\), \(a_{21} = c\,\), and \(a_{22} = d\). Thus \(T\) becomes

\[
x_1' = a x_1 + b x_2 \\
x_2' = c x_1 + d x_2
\]

and if \(a_{11} = -a_{22}\,\), \(T\) becomes
\[ x'_1 = ax_1 + bx_2, \]
\[ x'_2 = cx_1 - ax_2. \]

Now if we apply this transformation twice we have

\[
T^2 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} a^2 + bc & 0 \\ 0 & a^2 + bc \end{bmatrix}
\]

\[
\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} = I.
\]

Thus \( d = -a \) is the sufficient condition that the transformation \( T \) is an involution.

**Point Reflection.**—A point reflection is defined to be a dilation of period two and is expressed by the equations

\[
(2-7) \quad x' = -x + c, \quad y' = -y + c,
\]

which may be obtained from those of a dilation by requiring that the square of the matrix (2-6) of the dilation be the identity matrix. Under a point reflection, one point, say \( C \), remains fixed and all other points are reflected through this point, i.e., every point on the ideal line (axis) is invariant, every line on point \( C \) (center) is invariant, and, if \( P \) corresponds to \( P' \) under the point reflection, then \( C \) bisects the segment \( PP' \) and \( CP' = CP \). This relationship is shown in Figure 2.2.

If we consider the point \( C: (a, b) \), we can determine the center of the point reflection. It follows that

\[ a = x/2 + x'/2 \quad \text{and} \quad b = y/2 + y'/2, \]
which implies that $x + x' = 2a$ and $y + y' = 2b$. Let $2a = c_1$ and $2b = c_2$. Then we have

\begin{align*}
2-8) \quad & x + x' = c_1 \quad \Rightarrow \quad x' = -x + c_1,
& y + y' = c_2 \quad \Rightarrow \quad y' = -y + c_2.
\end{align*}

Hence, $a = c_1/2$, $b = c_2/2$. Therefore, the point $(c_1/2, c_2/2)$ remains fixed and is known as the center of the point reflection.

It is interesting to note that the product of two point reflections is a translation. We represent the point reflection in matrix form by

\begin{equation}
2-9) \quad \begin{bmatrix}
-1 & 0 & c \\
0 & -1 & c \\
0 & 0 & 1
\end{bmatrix}.
\end{equation}

Suppose we let (2-9) $A$ and let (2-10) $B$ which is a second point reflection given by

\begin{equation}
2-10) \quad \begin{bmatrix}
-1 & 0 & d \\
0 & -1 & d \\
0 & 0 & 1
\end{bmatrix}.
\end{equation}
If we form the product $AB$ we will have a matrix of the form

$$(2-11) \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix},$$

where $k$ and $k$ are constants. Comparing $(2-11)$ with $(2-3)$ we can see that $(2-11)$, which is the product of two point reflections, has the same form as $(2-3)$ which is a translation (Fig. 2.3).

![Figure 2.3](image)

Point reflection $A$ carries $Q \rightarrow Q'$ and $P \rightarrow P'$ and $B$ carries $Q' \rightarrow Q''$ and $P' \rightarrow P''$, where a translation, which moves a figure a certain distance in a given direction, carries $P \rightarrow P''$ and $Q \rightarrow Q''$. 
CHAPTER III

EQUIAREAL AND EQUIAFFINE TRANSFORMATIONS

The equiareal and equiaffine transformations form subsets of the group of affine transformations possessing a certain relationship to each other. To see this relationship, we define the set of equiareal transformations as being the set of affine transformations that preserves the numerical value of the area of a triangle, i.e., any affine transformation that preserves areas. Any equiareal transformation may be expressed in the form

\[(3-1) \quad x' = a_1x + b_1y + c_1, \quad y' = a_2x + b_2y + c_2 \quad (a_1, b_1, a_2, b_2, c_2 = \pm 1).\]

This set of transformations forms a group and has the set of equiaffine transformations as a subgroup. The set of equiaffine transformations is defined to be the set of affine transformations that preserves the measures of triangles, i.e., any affine transformation that preserves the magnitudes and signs of areas. We may express any equiaffine transformation in the form

\[(3-2) \quad x' = a_1x + b_1y + c_1, \quad y' = a_2x + b_2y + c_2 \quad (a_1, b_1 = a_2, b_2 = 1).\]

In our definition of an equiaffine transformation, we used the concept "measure of a triangle". We want to discuss this concept in relation to the transformations in
question. Consider three ordinary points \( A: (a, a_x, 1) \), 
\( B: (b, b_x, 1) \), and \( C: (c, c_x, 1) \). We know from our knowledge of collinear points that points \( A \), \( B \), and \( C \) are collinear if and only if the determinant

\[
\begin{vmatrix}
a & a_x & 1 \\
b & b_x & 1 \\
c & c_x & 1 \\
\end{vmatrix} = 0.
\]

We now let the determinant be denoted by \( m(A \ B \ C) \) and define the determinant as the measure of the triangle determined by the noncollinear points \( A \), \( B \), and \( C \). Therefore, three points \( A \), \( B \), and \( C \) form a triangle if and only if \( m(A \ B \ C) \neq 0 \). The sign of the measure of a triangle is determined by the order in which the vertices are named or the direction in which we transverse or orient the triangle. For example, 
\( m(A \ B \ C) = -m(A \ C \ B) \). From this relationship it follows that 
\( m(A \ B \ C) = m(B \ C \ A) = m(C \ A \ B) \)
and \( m(A \ C \ B) = m(C \ B \ A) = m(B \ A \ C) \).

The sign of the measure will become clearly evident if we consider three noncollinear points \( A \), \( B \), and \( C \) as in Figure (3.1) which by definition form a triangle. If we begin

![Figure 3.1](image-url)
with each point and transverse the triangle in a counterclockwise sense, naming each point in the order in which it is reached, we obtain ordered points which are said to be in the same cyclic order. Further, we are said to have transversed the triangle in a positive sense and the \( m(ABC) = m(BCA) = m(CAB) \) all of which are positive. Similarly, if we begin with each point and transverse the triangle in a clockwise sense, naming each point in the order in which it is reached, we obtain ordered points which are said to be in the same cyclic order, but of an order different from those obtained when transversed in a counterclockwise sense. Therefore, we are said to have transversed the triangle in a negative sense and the \( m(ACB) = m(CBA) = m(BAC) \) all of which are negative.

The magnitude or numerical value is related to the area of the triangle \( ABC \), i.e., the numerical value of \( m(ABC) \) is the area of triangle \( ABC \) in terms of a unit triangle. By comparing the orientation of triangle \( ABC \) with the unit triangle, we can determine the sign of the measure. Suppose we consider the triangle \( OIK \) (Fig. 3.2) in any real affine plane with a coordinate system, where \( OIK \) is determined by the points with homogeneous coordinates \((0, 0, 1), (1, 0, 1), \) and \((0, 1, 1, )\) respectively. Let \( OIK \) have measure 1 and be our unit triangle. Then the \( m(ABC) \) is positive if triangle \( ABC \) is transversed in the same sense as \( OIK \) and negative if
it is transversed in the opposite sense of OIK. Therefore, it is easily seen that two triangles ABC and A'B'C' given in Figure 3.3 are oppositely transversed or oriented.

The unit triangle OIK is transformed into a triangle of measure ±1 by an affine transformation if and only if the determinant of the transformation is +1. Generally, any triangle ABC with measure m(ABC) is transformed under an affine transformation with determinant D into a triangle A'B'C' with measure D[m(ABC)]. Thus an affine transformation perserves the measure of triangles if and only if it has determinant +1; it perserves the magnitude of the measure of triangles if and only if it has determinant of numerical value 1 [6;177-8]. It is on the basis of this information that we gave our definition of equiaffine and equiareal transformations.
CHAPTER IV

SIMILARITY TRANSFORMATIONS

Nature and Definition of Similarity Transformations.—

We shall obtain the group of euclidean transformations as a special subset of the group of affine transformations. To do this smoothly, we first consider the similarity transformation which is defined to be the affine transformation which leaves $I_\infty$ invariant, where $I_\infty$ is an elliptic involution. An involution is elliptic if $a^2 + bc$ is negative, where $a^2 + bc \neq 0$ is the condition which must be satisfied by the representation of any involution on a projective line which has the form

$$x_1' = ax_1 + bx_2,$$
$$x_2' = cx_1 - ax_2.$$

We shall define perpendicular lines in terms of an elliptic involution. Then the set of similarity transformations that are also equiareal transformations form the group of euclidean transformations under which parallelism, areas, distances, and angles are invariant.

Consider a projective plane $\mathcal{P}$ on which an ideal line $l_\infty$ is selected. Recall that the affine transformation leaves $l_\infty$ invariant as a line. We now place further restrictions upon the set of transformations under consideration by requiring that the pairs of an elliptic involution on $l_\infty$ be invariant.
Let $I_\infty$ be an arbitrary elliptic involution on $1_\infty$. The set of projective transformations leaving $I_\infty$ invariant forms a subgroup of the group of affine transformations and is called the group of similarity transformations. If a plane figure $F$ corresponds to a figure $F'$ under a similarity transformation, the two figures are said to be similar. If $l$ and $l'$ are two lines whose ideal points are a pair of the absolute involution, then the lines are said to be perpendicular or orthogonal. Thus a projective transformation that preserves parallelism is an affine transformation; an affine transformation that preserves perpendicularity is a similarity transformation.

Any similarity transformation leaves the ideal line invariant as a line and is an affine transformation

$$
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  0 & 0 & 1
\end{bmatrix},
$$

where $a_{11} - a_{12} a_{22} \neq 0$. By selecting the subset of the affine transformation under which the absolute involution is invariant, we shall obtain the equations of similarity transformations. The transformation (4-1) leaves the absolute involution

$$
\begin{align*}
x'_1 &= x \\
x'_2 &= -x
\end{align*}
$$

invariant if and only if every pair of points $(x_1, x_2, 0)$, $(x_2, -x, 0)$ of the involution corresponds under the
transformation (4-1) to a pair of the involution, i.e., if and only if the points

\[(a_1 x + a_2 x_2, a_2 x + a_2 x_2, 0)\]
\[(a_1 x_2 + a_2 x, a_2 x_2 - a_2 x, 0)\]

are a pair of the absolute involution. Since we are concerned with homogeneous coordinates, these points form a pair of the absolute involution if and only if there exist a real number \(k \neq 0\) such that for all \(x_1\) and \(x_2\)

\[a_1 x + a_2 x_2 = k(a_2 x_2 - a_2 x_1),\]
\[a_2 x_2 + a_2 x = -k(a_1 x_2 - a_1 x),\]

that is if and only if

\[(a_1 + ka_2)x + (a_2 - ka_1)x_2 = 0,\]
\[(a_2 - ka_2)x + (a_2 + ka_2)x_2 = 0.\]

Since these relations must hold for all \(x_1\) and \(x_2\), all four coefficients must vanish. The vanishing of the coefficients

\[a_1 = ka_2, \quad a_2 = ka_1,\]

implies that \(k^2 = 1\) and \(a_1^2 = a_2^2\). Similarly, the vanishing of the coefficients

\[a_2 - ka_2, \quad a_2 - ka_2,\]

implies that \(a_2^2 = a_2^2\). Finally, the existence of a single value of \(k \neq 0\) such that all four of the coefficients vanish implies that \(a_1, a_2 + a_2, a_2 = 0\). In other words, the points (4-3) are a pair of absolute involution and the transformation (4-1) leaves the absolute involution invariant if and only if

\[a_1^2 = a_2^2, \quad a_2^2 = a_2^2, \quad \text{and} \quad a_1, a_2 + a_2, a_2 = 0.\]
This result may also be expressed as follows: Any similarity transformation may be expressed in the form

\[ x' = ax - by + c, \]
\[ y' = bex + aey + d \quad (a^2 + b^2 \neq 0, e^2 = 1). \]

It is easily seen that the set of Euclidean transformations can be defined to be the set of similarity transformations which are also equiareal transformations. Under an equiareal transformation, any two figures that correspond have the same area. Any two figures that correspond under a similarity transformation are said to be similar and to have the same shape. Hence, the possibility of defining a Euclidean transformation as a similarity transformation that is also an equiareal transformation implies that any two figures that have the same shape and the same area are congruent in Euclidean geometry. We shall define Euclidean transformations in terms of orthogonal line reflections and show that such a definition is equivalent to the one given above.

**Orthogonal Line Reflections.**—A line reflection is a projective transformation of period two that leaves every point invariant on an ordinary line (axis) and every line invariant on an ideal point (center) that is not on the axis. An orthogonal line reflection is defined to be a line reflection having its center \( P \) and the ideal point \( P' \) on its axis as a pair of points of the absolute involution.
Intuitively a figure $F$ may be reflected in a line $m$ to obtain a figure $F'$ if the line segment joining the corresponding points of $F$ and $F'$ are all parallel and have their midpoints on the line $m$ (Fig. 4.1). If, in addition, the segments are perpendicular to the line $m$ (Fig. 4.2) the line reflection is called an orthogonal line reflection. Any orthogonal line reflection preserves shape and area.

We can express any line reflection algebraically in the form

\begin{align}
    x_1' &= a_{11} x_1 + a_{12} x_2 + a_{13} x_3,
    \\
    x_2' &= a_{21} x_1 + a_{22} x_2 + a_{23} x_3,
    \\
    x_3' &= x_3,
\end{align}

where

\begin{align*}
    a_{12} a_{23} + (a_{11} + 1) a_{13} &= 0, \\
    a_{21} a_{13} + (1 - a_{22}) a_{23} &= 0, \\
    a_{12} + a_{13} a_{22} &= 1.
\end{align*}

The line reflection (4-5) has center

\((-a_{12}, a_{11} + 1, 0)\)

and the point
as the ideal point on its axis. The line reflection is an orthogonal line reflection if and only if these two ideal points are a pair under the absolute involution (4-2), i.e., if and only if \( a_{/2} = a_{2/} \). Then, in nonhomogeneous coordinates, any orthogonal line reflection may be expressed in the form

\[
\begin{align*}
x' &= ax + by + c, \\
y' &= bx - ay + d,
\end{align*}
\]

where \( a^2 + b^2 = 1, (a + 1)c + bd = 0 = bc + (1 - a)d. \)

Any orthogonal line reflection is a similarity transformation and also an equiareal transformation. Any product of orthogonal line reflections is a similarity transformation (preserves shape) and also an equiareal transformation (preserves area). Any product of an even number of orthogonal line reflections preserves shape, area and signs of measure of ordered point triads (i.e., is an equiaffine transformation). Finally, any transformation that is both a similarity transformation and an equiareal transformation may be expressed as a product of orthogonal line reflections; any transformation that is both a similarity transformation and an equiaffine transformation may be expressed as a product of an even number of orthogonal line reflections. This property establishes the equivalence of the definitions of a euclidean transformation as a product of orthogonal line reflections and as a transformation that is both a similarity transformation and an equiareal transformation (i.e., as an affine
transformation that preserves shape and area) [6;193].

Intuitively, an orthogonal line reflection may be visualized in three-space as a 180° rotation of the plane about a line on the plane (the axis of reflection). This transformation can be performed as a rigid motion in space but cannot be performed as a rigid motion when the plane is considered by itself. We mean by rigid motion the moving of a configuration into another position, but making no change in its shape or size. In general, the similarity transformations that are also equiareal transformations are rigid motions when the plane is considered as a subset of three-space; the similarity transformations that are also equiaffine transformations are rigid motions when the plane is considered by itself. All transformations that are rigid motions when the plane is considered by itself are rigid motions when the plane is considered as a subset of three-space.

Now that we have discussed the nature of the orthogonal line reflection, we can prove the results which follow.

Result 1. Prove geometrically that the product of two orthogonal line reflections with parallel axes is a translation perpendicular to the axes.

Proof: Given two parallel axes m and m' and an orthogonal line reflection T which carries the figure AB onto A',B', i.e., T(AB) = A',B'. By the nature of an orthogonal line reflection, distance BL = B'L and AM = A,M. Given a second orthogonal line reflection T' performed on the results of T we get
Figure 4.3
figure $A_2B_2$, i.e., $T'T(AB) = A_2B_2$. By the same reasoning, $A_1M_1 = A_2M_2$ and $B_1L_1 = B_2L_2$. Since $m$ is parallel to $m'$ and we are considering orthogonal line reflections, $LL' = MM'$ and $AB = A_2B_2 = A_2B_2$. If we draw lines $Bb$, $Bb$, $Aa$ and $Aa$ parallel to lines $m$ and $m'$, then

$$BL = bM \quad (1),$$
$$LL' = MM' \quad (2),$$
$$L'a_z = M'A_z \quad (3).$$

From elementary geometry triangles $ABb$ and $AaBb$ are congruent. Hence,

$$Ab = a_zB_2 \quad (4).$$

Equations (1), (2), (3), and (4) and the congruence of triangles $ABb$ and $A_2B_2$ implies that $BB_2 = AA_2$ and $BB_2$ is parallel to $AA_2$. Therefore, $BA$ goes over to $B_2A_2$ which is a translation perpendicular to the axes. Hence, the proof is complete.

**Result 2.** Prove geometrically that the product of two orthogonal line reflections whose axes are perpendicular is a point reflection.

**Proof:** Given the perpendicular axes $m$ and $m'$, we choose an arbitrary point $P$ which is reflected onto point $P'$ by an orthogonal line reflection $T$ through axis $m$. By a second orthogonal line reflection $T'$, $P'$ is reflected onto $P''$. We shall prove that the product of these two orthogonal line reflections is a point reflection with center at the point of intersection of the two axes $m$ and $m'$ (Fig. 4.4). It is
sufficient to show that $P$, $O$ and $P''$ are collinear and $OP = OP''$. By definition of orthogonal line reflection $PL = LP'$ and $P'L' = L'P''$ and $PP'$ makes an angle $\beta'$ of 90° with $m$. Similarly, $P''P'$ makes an angle $\beta$ of 90° with $m'$. Hence, $OLL'P'$ form a rectangle and $\gamma = 90°$. Further, angle $\theta = \angle \phi'$. By elementary geometry, triangles $P''L'O$ and $OPL$ are congruent. Therefore, $OP = OP''$. Further,

$$\gamma = \gamma' \quad \theta + \gamma = 90° \quad \theta + \gamma' = 90° \quad \theta + \gamma' + \gamma = 180°$$

Therefore, $P''$, $O$, and $P$ are collinear. Thus the proof is complete.
Euclidean Transformations.--- We now define a euclidean transformation as the product of a finite number of orthogonal line reflections. Since the matrix of any orthogonal line reflection has determinant $-1$, we must classify euclidean transformations as odd or even according as they are obtained as a product of an odd or an even number of orthogonal line reflections. An even euclidean transformation has determinant $+1$ and is frequently called a displacement. An odd euclidean transformation has determinant $-1$.

The product of an even number of orthogonal line reflections may be expressed in the form

$$(1) \quad x' = ax - by + c,$$

$$y' = bx + ay + d \quad (a^2 + b^2 = 1).$$

When the plane is considered by itself rather than as a subset of three-space, the group of even euclidean transformations is often called the group of rigid motions. Therefore, euclidean geometry on a plane considered by itself is the geometry of the group of even euclidean transformations. Whereas, when the plane is considered as a subset of three-space, the odd euclidean transformation must be considered.

We have defined an odd euclidean transformation as being the product of an odd number of orthogonal line reflections. Any odd euclidean transformation may be expressed in the form

$$(2) \quad x' = ax + by + c,$$

$$y' = bx - ay + d \quad (a^2 + b^2 = 1).$$
Any even euclidean transformation may be considered as a rigid motion whether the plane is considered by itself or as a subset of three-space. Any odd euclidean transformation may be considered as a rigid motion when the plane is considered as a subset of three-space but is not considered as a rigid motion when the plane is considered by itself. This distinction between even and odd euclidean transformations is the distinction between the set of similarity transformations that are also equiaffine transformations and the set of similarity transformations that are also equiareal transformations but are not equiaffine transformations.

Any rigid motion is either a displacement (4-7) or an odd euclidean transformation (4-8). Accordingly, any rigid motion may be represented in the form

\[
x' = ax - by + c, \\
y' = bex + aey + d \quad (a^2 + b^2 = e^2 = 1).
\]

The transformation (4-9) represents an even transformation when \( e = +1 \), an odd transformation when \( e = -1 \).

In order to define euclidean geometry as a special case of affine geometry, we use the fact that distance is invariant under the group of euclidean transformations. Suppose we consider the general affine transformation given by

\[
x' = a_1x + b_1y + c_1, \\
y' = a_2x + b_2y + c_2,
\]

and consider the distance between two points \( P: (x_1, y_1) \) and \( Q: (x_2, y_2) \). Assume that the distance \( PQ \) remains invariant
under a euclidean transformation. Observe
\[ x'_1 = a_1 x + b_1 y + c_1, \]
\[ y'_1 = a_2 x + b_2 y + c_2, \]
and
\[ x'_2 = a_1 x + b_1 y + c_1, \]
\[ y'_2 = a_2 x + b_2 y + c_2. \]
Then
\[ (x'_2 - x'_1) = a_1 (x_2 - x_1) + b_1 (y_2 - y_1), \]
\[ (y'_2 - y'_1) = a_2 (x_2 - x_1) + b_2 (y_2 - y_1). \]
We define the distance \( PQ \) between any points \( P: (x_1, y_1) \) and \( Q: (x_2, y_2) \) to be the nonnegative square root of the expression
\[ (x_2 - x_1)^2 + (y_2 - y_1)^2, \]
where the \( x_j \) and \( y_j \) are nonhomogeneous coordinates of the points. That is to say,
\[ PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
and \( P'Q' = \sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2} \),
where \( P'Q' \) represents the transformed distance and \( PQ \) satisfies the following conditions:

(i) \( PQ = 0 \), if and only if \( P = Q \),

(ii) \( PQ = QP \), and

(iii) \( PQ + QR \geq PR \).

If this distance \( PQ \) is preserved then \( PQ \) must equal \( P'Q' \).
Observe
\[ \sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
\[ (x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \]
\[
\left[a_1(x_2 - x_1) + b_1(y_2 - y_1)\right]^2 + \left[a_2(x_2 - x_1) + b_2(y_2 - y_1)\right]^2
= (x_2 - x_1)^2 + (y_2 - y_1)^2
\]

\[
a_1^2(x_2 - x_1)^2 + b_1^2(y_2 - y_1)^2 + 2a_1b_1(x_2 - x_1)(y_2 - y_1)
+ a_2^2(x_2 - x_1)^2 + b_2^2(y_2 - y_1)^2 + 2a_2b_2(x_2 - x_1)(y_2 - y_1)
\]

This is to be true for all points \((x, y), (x', y')\). Thus

\[
a_1^2 + a_2^2 = 1 \quad (1)
b_1^2 + b_2^2 = 1 \quad (2)
a_1b_1 + a_2b_2 = 0 \quad (3)
\]

Considering (3), we have that

Set

\[
b_2/a_2 = k \text{ and } -b_1/a_1 = k.
\]

Then

\[
b_2 = a_1k,
b_1 = -a_1k.
\]

Substitute this results in (2).

\[
k^2 (a_2^2 + a_1^2) = 1.
\]

But from (1) \(a_2^2 + a_1^2 = 1\). Therefore, \(k^2 = 1 \rightarrow k = \pm 1\).

If \(k = \pm 1\), then \(b_2 = a_1^2\) and \(b_1 = a_1^2\). Thus

\[
a_1^2 + b_1^2 = a_1^2 + a_2^2 = 1
\]

and

\[
a_2^2 + b_2^2 = a_1^2 + a_2^2 = 1.
\]
Therefore,

\[ x' = a_{1}x - a_{2}ky + c_{1}, \]
\[ y' = a_{2}x + a_{1}ky + c_{2}, \]

where
\[ a_{1}^{2} + a_{2}^{2} = 1 \text{ and } k = \pm 1. \]

If \( k = 1 \) the transformation is even and

\[ x' = a_{1}x - a_{2}y + c_{1}, \]
\[ y' = a_{2}x + a_{1}y + c_{2} \quad (a_{1}^{2} + a_{2}^{2} = 1). \]

Then we can write
\[ a_{1} = \cos \theta \]
\[ a_{2} = \sin \theta \]

since \( \sin^{2} \theta + \cos^{2} \theta = 1 \). Thus we have the euclidean transformation

\[(4-7) \quad x' = x \cos \theta - y \sin \theta + c_{1}, \]
\[ y' = x \sin \theta + y \cos \theta + c_{2}. \]

If \( c_{1} = c_{2} = 0 \), then the euclidean transformation becomes

\[ x' = x \cos \theta - y \sin \theta \]
\[ y' = x \sin \theta + y \cos \theta \]

which is known as a rotation about the origin \((0, 0)\) which remains fixed.

If \( \cos \theta = 1 \) and \( \sin \theta = 0 \), then the euclidean transformation becomes

\[ x' = x + c_{1}, \]
\[ y' = y + c_{2}, \]

which represents a translation.
If we set \( \theta = \pi = 180^\circ \), then the euclidean transformation becomes

\[
\begin{align*}
x' &= -x + c_1, \\
y' &= -y + c_2,
\end{align*}
\]

which represents a point reflection.

**Angles and Directed Angles**— Any point 0 of an ordinary line in a euclidean plane divides the line into two parts each of which is called a ray (or half-line) issuing from 0. The two rays issuing from a point 0 on the same side are said to be opposite. The figure formed by two rays issuing from a point 0 is called an angle; the point 0 is called the vertex and the rays are called the sides of the angle [10;173].

Suppose the two rays are represented generally by \{BA\} and \{BC\} where \( B: (b_1, b_2) \) is the common initial point. We then have an angle ABC with vertex B and sides \{BA\} and \{BC\}. Then given any angle ABC, there exist a unique rotation of the side \{BA\} onto the side \{BC\} and a second unique rotation of the side \{BC\} onto the side \{BA\}. We can indicate a particular rotation by specifying the rays in a particular order and requiring that the first be rotated onto the second. Any ordered pair of rays will form a figure with a common initial point or vertex, an initial side and a terminal side which is called a directed angle.

We now see the relationship between directed angles and rotations. Equivalent rotations are rotations in the same
sense and of the same amount that differ at most in their constant terms. We say that two directed angles are associated with two equivalent rotations, i.e., directed angles are equal if and only if they correspond under a displacement.

An undirected angle AOB may be associated with both the rotation of \{OA\} onto \{OB\} and the rotation of \{OB\} onto \{OA\}. Thus angle AOB is associated with an amount of rotation without regard to the sense of the rotation.

The special cases of affine transformations that we have considered are indicated in the following array. The lines are used to indicate that a set of transformations has been obtained from the one above it by placing certain restrictions on the latter. In other words, a set of transformations is a subset of the set above it. In this way, we are able to see the existing relationship at a glance.

\[
\begin{align*}
\text{Affine} & \quad \text{Orthogonal} \\
\text{Equiareal} & \quad \text{Simmetry} & \quad \text{Equihomeotic} \\
\text{Equi-affine} & \quad \text{Homothetic} & \quad \text{Dilation} \\
\text{Orthogonal} & \quad \text{Translation} & \quad \text{Point Reflection} \\
\text{Euclidean} & \quad \text{Identity} & \quad \text{Identity} \\
\end{align*}
\]


