MAPPING CERTAIN FUNCTIONS

A THESIS

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-Nellie Wolfe Warrington
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CHAPTER I

INTRODUCTION

Real functions of real variables, \( y = f(x) \), can be represented graphically by plotting corresponding values of \( x \) and \( y \) as rectangular coordinates of points in the XY - plane. When the variables are complex, the variables \( z \) and \( w \) are exhibited geometrically by points in the complex plane. It is generally simpler to use separate planes for the two variables, \( z \) - points in the \( Z - \) plane and the \( w \) - points in the \( W - \) plane. Then corresponding to each point \((x,y)\) in the \( Z - \) plane for which \( f(x \pm iy) \) is defined, there will be a point \((u,v)\) in the \( W - \) plane, where \( w = u \pm iv \). That is, the function \( f(z) \) maps points in the \( Z - \) plane upon points in the \( W - \) plane. The correspondence between the points in the two planes is called a MAPPING of points in the \( Z - \) plane into points in the \( W - \) plane by the function \( w = f(z) \).

To do the mapping in this treatise we must understand the terms used and we must know from whence our facts come. This information will be located in Chapter II, which supplies us with definitions and theorems with proofs; such as complex numbers and its geometrical representation, addition and subtraction of complex numbers, the product and quotient of complex numbers, fundamental definitions concerning functions, Cauchy-Goursat theorems, Cauchy's integral formula theorems, theorems on Cauchy-Riemann differential equations, Laplace's differential equations, and
Demoivre's theorem. The writer makes use of the last two theorems frequently in discussing certain functions in which the first three theorems are needed to discuss the Laplace's differential equation theorem.

Chapter III gives a discussion of Mapping in general with applications to certain functions, $w = z^2$, $w = z^n$, and $w = \log z$. Each function has a definite mapping and will be seen geometrically. The function $w = z^2$ maps a portion of one plane upon a portion of another plane, the function $w = z^n$ maps a portion of one plane upon an entire plane and $w = \log z$ maps the whole of one plane upon another.

Chapter IV deals with the application of mapping by means of some definite functions.
CHAPTER II
DEFINITIONS AND THEOREMS

Complex Numbers and Geometrical Representation. — If $a$ and $b$ are any two real numbers and $i^2 = -1$, $a + bi$ is called a complex number and $a - bi$ its conjugate. Two complex numbers $a + bi$ and $c + di$ are said to be equal if and only if $a = c$ and $b = d$. If $b \neq 0$, $a + bi$ is said to be imaginary and $bi$ is called a pure imaginary.

Using rectangular axes of coordinates, $OX$ and $OY$, we represent the complex number $a + bi$ by the point $P$ having the coordinates $a, b$ (Figure 1, Page 4).

The positive number $\rho = \sqrt{a^2 + b^2}$ giving the length of $OP$ is called the modulus (or absolute value, $|\rho|$) of $a + bi$. The angle $\theta = PQA$, measured counter-clockwise from $OA$ to $Op$, is called the amplitude (or argument) of $a + bi$. The angle $\theta$ lying within the interval $-\pi \leq \theta \leq \pi$ is called the chief amplitude.

Finding the cos and sin of $\theta$, we get

$$\cos \theta = \frac{a}{\rho},$$
$$\sin \theta = \frac{b}{\rho},$$

where $a = OA$, $b = OB$,

whence $a + bi = \rho \left( \cos \theta + i \sin \theta \right)$.

Two complex numbers are equal if and only if their moduli are equal and an amplitude of the one is equal to an amplitude of the other or an amplitude of the other increased by a multiple of $2\pi$. 

3
Addition and Subtraction of Complex Numbers. - Addition of complex numbers is defined by

\[(a + bi) + (c + di) = (a + c) + i(b + d).\]

The inverse operation to addition is called subtraction, and consists in finding a complex number \(z\) such that

\[(c + di) + z = a + bi.\]

In notation and value, \(z\) is

\[(a + bi) - (c + di) = (a - c) + i(b - d).\]

Product of Complex Numbers. - By actual multiplication

\[\rho (\cos \theta + i \sin \theta) \rho' (\cos \phi + i \sin \phi) \]

by trigonometry. Hence

the modulus of the product of two complex numbers is equal to the product of their moduli, while the amplitude of the product is equal to the sum of their amplitudes.

Taking \(\rho = \rho' = 1\) in the above relation, we obtain the useful formula

\[\cos \theta + i \sin \theta \left( \cos \phi + i \sin \phi \right) = \cos(\theta + \phi) + i \sin(\theta + \phi).\] (1)

Division of Complex Numbers. - Let \(\phi = \beta - \theta\) in (1) and divide the members of the resulting equation by \(\cos \theta + i \sin \theta\), we get

\[
\frac{\cos \phi + i \sin \phi}{\cos \theta + i \sin \theta} = \cos(\beta - \theta) + i \sin(\beta - \theta)
\]

which is defined as quotient of complex numbers. Hence the amplitude of the quotient of \(R (\cos \phi + i \sin \phi)\) by \(r (\cos \theta + i \sin \theta)\) is equal to the difference \(\beta - \theta\) of their amplitudes, while the modulus of the quotient is equal to the quotient \(R/r\) of their
moduli.

The case $\beta = 0$ gives the useful formula

$$\frac{1}{\cos \theta + i \sin \theta} = \cos \theta - i \sin \theta.$$  

**Fundamental Definitions Concerning Functions.** We shall make the same distinction between constants and variables as in the realm of real variables. If a complex number assumes but a single value in any discussion, it is called a **constant**. If, on the other hand, a complex number is allowed in any discussion to assume various complex values, it is called a **variable**. A complex variable $z$ may be written in the form $x + iy$, where $x$ and $y$ are real variables.

Let us speak of a connected portion of the complex plane as a **region**. Any point $z_0$ is said to be an **inner point** of a region if it can be made the center of a circle of radius different from zero such that all points within this circle are points of the region. If the circle can be taken so small that it contains no points of the region, then $z_0$ lies exterior to the region. If the circle contains both points of the region and points exterior to it, however small the radius of the circle be taken, then $z_0$ is called a **boundary point** of the region. If the boundary is included in the region, then it is spoken of as a **closed region**. If the boundary is not included in the region, it is called an **open region**.

The following definition of a function of a real variable, which is the one commonly accepted by mathematicians at the present time was formulated by Dirichlet in 1837 from his study of
Fourier's theory of heat (7:21).

If for each value of a variable \( x \), there is determined a definite value or set of values of another variable \( y \), then \( y \) is called a function of \( x \) for those values of \( x \).

What is essential in the definition is that for every value that \( x \) does take, there is thereby determined a definite value or definite values of \( y \). From this idea Cauchy gave to the variable complex values and extended the notion of a definite integral by letting the variable pass from one limit of integration to the other through a succession of complex values along arbitrary paths.

We shall understand the complex variable \( w \) to be a function of the complex variable \( z \) in a given open or closed region \( S \) if for each value of \( z \) in this region \( w \) has a definite value or set of values. Unless otherwise stated, it will be understood that \( z \) takes all values of a given region. Then \( w \) is a function of \( z \), we may write

\[
    w = u(x,y) + iv(x,y) = f(z)
\]

where \( u \) and \( v \) are real functions of the two variables \( x \) and \( y \). If \( w \) has but one value for each value of \( z \), \( w \) is said to be SINGLE-VALUED; if it takes two or more values for some or all of the values of \( z \), then \( w \) is called a MULTIPLE-VALUED function of \( z \).

A single-valued function \( f(z) \) is ANALYTIC at a point \( z_0 \) if and only if its derivative exists at every point in some neighborhood of \( z_0 \). When it exists, the derivative has, of course, a
Two real functions \( u \) and \( v \), which satisfy these conditions

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x},
\]

are called CONJUGATE FUNCTIONS.

If a given single-valued function \( f(z) \) has a uniquely determined derivative at the point \( \omega \) and at every point in the neighborhood for \( \omega \), then \( z = \omega \) is called a REGULAR POINT of \( f(z) \).

A point in every deleted neighborhood of which there are regular points but which is itself not a regular point is called a SINGULAR POINT of the given function. In referring to the points of a neighborhood of \( z_0 \) exclusive of the point \( z_0 \) itself, we shall speak of the region as DELETED NEIGHBORHOOD of \( z_0 \).

If every point of a region \( S \) is a regular point of a single-valued function \( f(z) \), then \( f(z) \) is said to be HOLOMORPHIC in \( S \).

Lines which meet at a point to form right angles are called ORTHOGONAL LINES.

If \( \omega \) is a limiting point of \( z \), and if corresponding to an arbitrarily small positive number \( \varepsilon \) there exists a positive number \( \delta \) such that for all values of \( z \) entering into the discussion for which \( |z - \omega| < \delta \), with the possible exception of \( z = \omega \), we have

\[
|f(z) - A| < \varepsilon,
\]

then \( f(z) \) is said to have the limiting value \( A \) corresponding to the limit \( \omega \) of \( z \). We may write the existence and value of this limit

\[
\lim_{z \to \omega} f(z) = A,
\]
9

which is defined as a LIMIT OF A FUNCTION.

A function of a complex variable is said to be CONTINUOUS AT A POINT if the value of the function at that point is equal to the limit of the values assumed by the function in every neighborhood of the point. There are three things involved in continuity; first, the function must be defined at the point in question; second, the functional value at the point must exist; third, the two must be equal.

A function is continuous on the right of \( x_0 \) if the limit of the values of the function on the right of the limiting point \( x = x_0 \) is the same as the value of the function at the point and may be written

\[
\lim_{x \to x_0} f(x) = f(x_0).
\]

Cauchy-Goursat Theorems.- In order to establish this theorem, the following lemma will be of use.

Lemma. Given a region \( T \) in which \( f(z) \) is holomorphic. Let the ordinary closed curve \( C \), lying wholly in \( T \) and containing only points of \( T \), be the complete boundary of a region \( T' \). It is always possible to divide the region \( T' \) into a finite number of squares \( S_i \) and partial squares \( R_i \) such that within or upon the boundary of each of these subregions there exists a point \( z_i \) such that as \( z \) describes the boundary of the subregion we have

\[
\left| \frac{f(z) - f(z_i)}{z - z_i} - f'(z_i) \right| < \epsilon,
\]

(1)
where $\varepsilon$ is a previously assigned arbitrarily small positive number.

We shall now demonstrate the Cauchy-Goursat theorems.

Theorem I. Let $f(z)$ be holomorphic in a given finite region $S$ and let $C$ be the complete boundary of any portion $S'$ of $S$ such that $C$ lies wholly in $S$ and incloses only points of $S$; then

$$\int_C f(z) \, dz = 0.$$  

Proof: The boundary $C$ may consist of a single closed ordinary curve or a combination of such curves. We shall first consider the case where $C$ is a single closed ordinary curve and the inclosed region is simply connected. Let $S'$ be divided into squares and partial squares. Let $n$ denote the number of squares $S_i$ and $m$ the number of partial squares $R_i$. If the integral is taken in a positive direction around the perimeter of the various squares $S_i$, $R_i$, each side of these regions that is not a portion of $C$ is taken twice as a path of integration, the two integrals being taken however in opposite directions. Considering the sum of the integral about the perimeters of all the regions $S_i$, $R_i$, we may therefore write

$$\int_C f(z) \, dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z) \, dz + \sum_{i=1}^{m} \int_{\lambda_i} f(z) \, dz,$$  

where $\gamma_i, \lambda_i$ denote the boundaries of $S_i, R_i$ respectively (Fig. 2, Page 11)

From the lemma, we have within or upon the boundary of each $S_i, R_i$ a point $z_1$ such that
\[ \left| \frac{f(z) - f(z_i)}{z - z_i} - f'(z_i) \right| \leq \varepsilon \]  

The relation can be written

\[ \frac{f(z) - f(z_i)}{z - z_i} = f'(z_i) + \varepsilon \]

where \( \varepsilon \) is a function of \( z \) such that \( \varepsilon \) as \( z \) varies along the contour of \( S_i \) or \( R_i \). We shall now consider the integral around the perimeter of one of the squares \( S_i \). Since \( z_i \) is a constant for this integral, we have from (4)

\[ \int_{\gamma_i} f(z) \, dz = \left[ f(z_i) - z_i f'(z_i) \right] \int_{\Gamma_i} f \, dz + f'(z_i) \int_{\Gamma_i} z \, dz + \int_{\gamma_i} \eta_i(z - z_i) \, dz. \]  

We know that

\[ \int_{\gamma} \, dz = r - \theta \quad \int_{\gamma} z \, dz = \frac{r^2}{2} \quad \int_{\theta} \, dz = \frac{r^2 \theta^2}{2}. \]

Taking the particular case under consideration, as the path of integration is a closed curve, \( \theta \) and \( \phi \) are the same point, and hence both of these integrals vanish. From equation (5) we then have

\[ \left| \int_{\gamma_i} f(z) \, dz \right| = \left| \int_{\gamma_i} \eta_i(z - z_i) \, dz \right|. \]

Let the length of one side of the square \( S_i \) be \( c_i \) (Figure 3).
As you can see the diagonal of the square is
\[ \sqrt{2c_1^2} \text{ or } c_1 \sqrt{2} \]. Hence, we have
\[ |z - z_i| = c_1 \sqrt{2} \]. (7)
Making use of this relation and the relation we may now write
\[ \left| \int_{c_i} f(z) \, dz \right| < \varepsilon c_1 \sqrt{2} \int_{c_i} |dz| = \varepsilon c_1 \sqrt{2} \times 4 c_1 = \varepsilon 4 \sqrt{2} A_i \]. (8)
where \( \varepsilon c_1 \sqrt{2} \) is the maximum value of \( |f(z)| \) along the path of integration, \( 4 c_1 \) is the length of the path of integration and \( A_i \) denotes the area of \( S_i \).
Consider now the integral taken around one of the partial squares. We have
\[ \int_{c_i} f(z) \, dz = \int_{c_i} f(z_i) \, dz + \int_{S_i} (z - z_i) \, dz \]. (9)
As before the first two integrals vanish in the second member of this equation and we have
\[ \left| \oint_{\Gamma} f(z) \, dz \right| = \left| \int_{\Gamma} (z - z') \, dz \right|. \]  

(10)

We may denote the length of a side of the square of which \( R_1 \) is a portion by \( c_i \) where \( R_1 \) is that portion of the square cut off by curve \( C \) and lying in \( S' \). Let \( l_1 \) be the length of that arc of \( C \) which forms a portion of the boundary of \( R_1 \). We have then

\[ \left| Z - Z_1 \right| = c_i \sqrt{2}. \]

From (10), we have

\[ \left| \sum_{\Gamma} f(z) \, dz \right| \leq \varepsilon c_i \sqrt{\frac{2}{c_i}} \left| \sum_{\Gamma} dz \right| \leq \varepsilon c_i \sqrt{\frac{2}{c_i}} (4A + c_i), \]

(11)

where \( B_1 \) denotes the area of the square of which \( R_1 \) is a part, and \( c \) is the length of one side of the largest square that comes into consideration in the subdivision of \( S' \).

Replacing each term of the sums in (2) by its absolute value, we have, by use of (8) and (11),

\[ \left| \sum_{\Gamma} f(z) \, dz \right| \leq \varepsilon \sqrt{\frac{2}{c_i}} \left( \frac{4A}{c_i} + \sum_{\Gamma} (4B_i + c_i) \right) \leq \varepsilon \sqrt{\frac{2}{c_i}} \left( 4A + c_iL \right), \]

where \( L \) denotes the length of the curve \( C \), and \( A \) denotes the combined area of the system of congruent squares with which the region \( S' \) was originally covered. The expression \( \left\{ 4A \neq CL \right\} \) is therefore a constant, and as \( \varepsilon \) is arbitrarily small the product is arbitrarily small. As the absolute value of the \[ \int_{\Gamma} f(z) \, dz \] is shown to be less than an arbitrarily small number, it follows, that

\[ \int_{\Gamma} f(z) \, dz = 0. \]

Theorem II. If \( f(z) \) is holomorphic in a finite closed multiply connected region \( S \) bounded by an exterior curve \( C \) and
a finite number of inner curves \( c_1, c_2, \ldots, c_n \), then
\[
\int_c f(z) \, dz = \sum_{k=1}^n \int_{c_k} f(z) \, dz
\]
each integral being taken in a positive direction with respect to the region inclosed.

Proof: Connect each inner curve \( c_k \) with the exterior curve \( C \) by a cross-cut thus making the region simply connected (Figure 4).

![Figure 4.](image)

From Theorem I, the integral taken around the complete boundary, including the cross-cuts, is zero. Since this integral is equal to the integral taken over the boundary of a multiply connected region, this follows
\[
\int_c f(z) \, dz + \sum_{k=1}^n \int_{c_k} f(z) \, dz = 0.
\]

If now the integrals along the curves \( c_k \) are taken in a positive direction with respect to the regions interior to these
curves rather than to the region $S$, the direction in which each integral is taken is changed and therefore we have

$$\int_{C} f(z) \, dz = \sum_{k=1}^{n} \int_{C_k} f(z) \, dz.$$  

Cauchy's Integral Formula.

**Theorem III.** Given a finite closed region $S$ whose boundary $C$ consists of a finite number of ordinary curves. If $f(z)$ is holomorphic within $S$ and converges uniformly along $C$, or if it is also holomorphic for values along $C$, then for any inner point $\phi$ of $S$ we have

$$f(\phi) = \frac{1}{2\pi i} \oint_{C} \frac{f(z) \, dz}{z - \phi}.$$  

Proof: In accordance with the statement of the theorem, the boundary $C$ may consist of one or more curves. Let $f(z)$ be single-valued and analytic in a region including the point $z = \phi$ and bounded by a curve $C$. Draw a small circle about the point $\phi$ with radius $r$ lying entirely within the region $S$ and inclosing only points of $S$. That portion of the region lying outside the inner circle, denote by $S'$ (Figure 5.)
Then in the area bounded by C and this circle the function \( f(z) \) is holomorphic in the region \( S' \), since by hypothesis \( f(z) \) is holomorphic in the region \( S \). Integrating about the contour of \( S' \)

\[
\int_C \frac{f(z)}{z-a} \, dz + \int_\gamma \frac{f(z)}{z-a} \, dz = 0,
\]

where the second integral is taken around the circle in a positive direction with respect to the region inclosed by the same circle. We have after transposing this integral to the second member of the equation

\[
\int_C \frac{f(z)}{z-a} \, dz = \int_\gamma \frac{f(z)}{z-a} \, dz.
\]  

(1)

Since \( f(z) \) is continuous at \( z = a \), we have

\[
\lim_{z \to a} f(z) = f(a);
\]

that is, for an arbitrarily small positive number \( \epsilon \), there exists a positive number \( \delta \) such that

\[
|f(z) - f(a)| < \epsilon
\]

for \( |z - a| < \delta \) or \( f(z) = f(a) \) \( \forall \epsilon \).

From equation (1) and substituting, we now have the integral

\[
\int_\gamma \frac{f(z)}{z-a} \, dz = f(a) \int_\gamma \frac{dz}{z-a} \neq \int_\gamma \frac{dz}{z-a}.
\]

(2)

On the circumference of the circle we have

\[
z - a = \gamma (\cos \theta + i \sin \theta),
\]
but \( \cos \theta / i \sin \theta = e^{i\theta} \), we then get
\[
\begin{align*}
z - \phi &= r e^{i\theta} \\
dz &= i r e^{i\theta} d\theta \\
dz &= i (z - \phi) d\theta \\
\frac{dz}{z - \phi} &= i d\theta.
\end{align*}
\]

As \( z \) describes the circle, \( \theta \) passes from 0 to \( 2\pi \), we have then
\[
\int_{\theta} \frac{f(z)}{z - \phi} \, dz = f(\phi) \int_{0}^{2\pi} d\theta + i \int_{0}^{2\pi} e^{i\theta} d\theta
\]
and letting the \( i \int_{0}^{2\pi} e^{i\theta} d\theta = \rho \)

therefore,
\[
\int_{\theta} \frac{f(z)}{z - \phi} \, dz = 2\pi i f(\phi) / \rho.
\]

Now we may take the radius of the circle so small that \( |\epsilon| \) is less than any assigned value for points on the circle. Hence \( |\rho| \) is less than any assigned value, and the value of
\[
\int_{\theta} \frac{f(z)}{z - \phi} \, dz
\]
differs from \( 2\pi i f(\phi) \) by a quantity which can be made as small as we please. Therefore, from (1)
\[
\int_{\theta} \frac{f(z)}{z - \phi} \, dz = 2\pi i f(\phi) \\
\]
\[
f(\phi) = \frac{1}{2\pi i} \int_{\theta} \frac{f(z) \, dz}{z - \phi}.
\]
Theorem IV. If \( f(z) \) is holomorphic in a given finite region \( S \), then the derivative \( f'(z) \) is a continuous function in \( S \); moreover, \( f'(z) \) is itself holomorphic in \( S \).

Proof: Let \( z_0 \) be an inner point of \( S \) and let \( C \) be an ordinary closed curve (Fig. 6). From Theorem III, we have

\[
f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(t) \, dt}{t - z_0},
\]

where \( t \) is a complex variable taken along the contour \( C \).

Let \( z_0 \neq \Delta z \) be any second point in the neighborhood of \( z_0 \) lying within the inner circle lying within \( C \) and having \( z_0 \) as a center and \( \Delta \) as a radius. We have then

\[
\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i} \oint_C \frac{f(t) \, dt}{(t-z_0-\Delta z)\Delta z} - \frac{1}{2\pi i} \oint_C \frac{f(t) \, dt}{(t-z_0)\Delta z}
\]

\[
= \frac{1}{2\pi i} \oint_C \frac{f(t) \, dt}{(t-z_0-\Delta z)(t-z_0)\Delta z}
\]

\[
= \frac{1}{2\pi i} \oint_C \frac{f(t) \, dt}{(t-z_0-\Delta z)(t-z_0)}.
\]

(6)
However,
\[
\frac{1}{(t - z_0)^2} \leq \frac{(t - z_0 - \Delta z)^2}{(t - z_0)^2 (t - z_0 - \Delta z)},
\]
and consequently,
\[
\int_{C} \frac{f(t) \, dt}{(t - z_0)^2} \leq \int_{C} \frac{f(t) \, dt}{(t - z_0)^2} \leq \int_{C} \frac{\Delta z \, f(t) \, dt}{(t - z_0)^2 (t - z_0 - \Delta z)}.
\]

We can readily show that the last of these integrals has the limit zero as \( \Delta z \to 0 \). To do so, let \( r \) be the lower limit of the distance of any point within \( C \) from a point on \( C \). We have then
\[
|t - z_0 - \Delta z| > r, \quad |t - z_0| > r.
\]

We may now write
\[
\left| \int_{C} \frac{\Delta z \, f(t) \, dt}{(t - z_0)^2 (t - z_0 - \Delta z)} \right| \leq \frac{M L}{2 r} |\Delta z|,
\]
where \( M \) is the maximum value of \( |f(t)| \) along \( C \) and \( L \) is the length of the curve \( C \). Hence as \( \Delta z \to 0 \), we have zero as the limit of this integral.

Consequently, passing to the limit as \( \Delta z \to 0 \), we have from (7)
\[
\lim_{\Delta z \to 0} \int_{C} \frac{f(t) \, dt}{(t - z_0)(t - z_0 - \Delta z)} = \int_{C} \frac{f(t) \, dt}{(t - z_0)^2}.
\]
Hence, from (6) we get
\[
\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2 \pi i} \int_{C} \frac{f(t) \, dt}{(t - z_0)^2},
\]
or
\[
f'(z_0) = \frac{1}{2 \pi i} \int_{C} \frac{f(t) \, dt}{(t - z_0)^2}.
\]
We may show in the same way that

\[ f'''(z_0) = \frac{2}{2\pi i} \int_C \frac{f(t) dt}{(t - z_0)^3}, \]

\[ f''''(z_0) = \frac{3}{2\pi i} \int_C \frac{f(t) dt}{(t - z_0)^4}, \]

\[ \ldots \ldots \ldots \]

\[ f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(t) dt}{(t - z_0)^{n+1}}. \]

The existence of these integrals enable us to affirm the existence of higher derivatives of \( f(z) \). Consequently, the derivative \( f'(z) \) is continuous and holomorphic as the theorem states.

**Theorem V.** If \( f(z) \) is continuous in a given region \( S \) and if the \( \int f(z) dz \) is zero when taken around the complete boundary \( C \) of any portion of \( S \), such that \( C \) lies wholly within \( S \) and incloses only points of \( S \), then \( f(z) \) is holomorphic in \( S \).

**Proof:** If the given region \( S \) is multiply connected, let it be made simply connected by the introduction of cross-cuts. Then every closed curve \( C \) lying within the new region \( S' \) is a complete boundary and by hypothesis the integral taken along such a curve is zero. We shall show that in this simply connected region \( f(z) \) is holomorphic and hence holomorphic in \( S \), even though this given region is multiply connected. Let \( \mathcal{S} \) be a fixed point of \( S \) and \( z_0 \) any other point of the same region. Denote by \( z_0 \neq Az \) any point of \( S' \) in the neighborhood of \( z_0 \). Because

\[ \int_C f(z) dz = 0, \]
and we are at liberty to select arbitrarily the path of integration between \( \beta \) and \( z_0 \neq \Delta z \), without affecting the value of the integral

\[
\oint_{\beta} f(z)dz,
\]

we take a path passing through \( z_0 \) and rectilinear between \( z_0 \) and \( z_0 \neq \Delta z \) (Figure 7).

\[
\text{Figure 7}
\]

The value of the integral \( \oint f(z)dz \), where \( \gamma \) is a point upon this path, is a function of \( \gamma \), and we may write

\[
F(\gamma) = \oint f(z)dz.
\]

Hence, we have for \( \gamma = z_0 \) and \( \gamma = z_0 \neq \Delta z \) the following values of \( F(\gamma) \):

\[
F(z_0) = \oint f(z)dz,
\]

\[
F(z_0 \neq \Delta z) = \oint \left( f(z) \right) d\gamma,
\]

whence

\[
\Delta F(z_0) = F(z_0 \neq \Delta z) - F(z_0) = \oint f(z)dz - \oint f(z)dz
\]

\[
= \oint f(z)dz.
\]

The given function \( f(z) \) is continuous in \( S \), and, therefore, we have for an arbitrarily small positive number \( \varepsilon \), another positive number \( \delta \) such that

\[
|f(z) - f(z_0)| < \varepsilon, \text{ for } |z - z_0| = \delta.
\]
This relation may be written

\[ f(z) = f(z_0) + \eta(z), \]

where \( |\eta| < \varepsilon \) for \( |\Delta z| < \delta \).

Putting \( f(z_0) + \eta \) in place of \( f(z) \) we obtain from (8)

\[ \Delta F(z_0) = \int_{z_0}^{z_0 + \Delta z} [f(z_0) + \eta] \, dz, \]

\[ \Delta F(z_0) = \int_{z_0}^{z_0 + \Delta z} f(z_0) \, dz + \int_{z_0}^{z_0 + \Delta z} \eta \, dz. \] (9)

Dividing by \( \Delta z \), we have

\[ \frac{\Delta F(z_0)}{\Delta z} = \int_{z_0}^{z_0 + \Delta z} \frac{f(z_0)}{\Delta z} \, dz + \int_{z_0}^{z_0 + \Delta z} \frac{\eta}{\Delta z} \, dz. \]

But

\[ \frac{1}{\Delta z} \int_{z_0}^{z_0 + \Delta z} f(z_0) \, dz = \frac{f(z_0)}{\Delta z} \int_{z_0}^{z_0 + \Delta z} d\overline{z}, \]

which is equal to

\[ \frac{f(z_0)}{\Delta z} \int_{z_0}^{z_0 + \Delta z} \left[ f(z_0) - f(z_0) \right] = \frac{f(z_0)}{\Delta z} (z_0 + \Delta z - z_0) \]

\[ = \frac{f(z_0)}{\Delta z} \cdot \Delta z \]

\[ = f(z_0). \]

Moreover, we have

\[ \left| \frac{1}{\Delta z} \int_{z_0}^{z_0 + \Delta z} \eta \, dz \right| \leq \frac{1}{|\Delta z|} \int_{z_0}^{z_0 + \Delta z} |\eta| \cdot |dz| < \frac{\varepsilon}{|\Delta z|} \int_{z_0}^{z_0 + \Delta z} |dz| = \varepsilon. \]

From (9) we now get

\[ \left| \frac{\Delta F(z_0)}{\Delta z} - f(z_0) \right| < \varepsilon, \quad |\Delta z| < \delta; \]

that is,

\[ \lim_{\Delta z \to 0} \frac{F(z_0)}{\Delta z} = F'(z_0) = f(z_0). \]
Since $z_0$ was taken to be any point of $S$, it follows that for all values of $z$ in $S$ we have

$$f'(z) = f(z).$$

(10)

Consequently, $F(z)$ is holomorphic in $S$; but, as we have seen, the derivative of such a function is also holomorphic; hence $f(z)$ must likewise be holomorphic in $S$, as the theorem requires.

The Cauchy-Goursat theorems state a necessary condition that a given function $f(z)$ shall be holomorphic in a given region. The theorem just demonstrated gives a sufficient condition that a continuous function is holomorphic. We may combine these two results into the following theorem.

Theorem VI. The necessary and sufficient condition that a continuous function $f(z)$ is holomorphic in a given finite region $S$ is that the integral $\int f(z)dz$ is zero when taken along the complete boundary $C$ of any portion of the plane when $C$ lies entirely within $S$ and incloses only points of $S$.

Cauchy-Riemann Differential Equations. We discussed the necessary and sufficient condition that a function $f(z)$ is holomorphic in a given finite region $S$ in the preceding discussion. This condition was expressed in terms of a definite integral. It is often more convenient to have such a condition expressed in terms of the partial derivatives of $u$ and $v$, where

$$f(z) = w = u(x,y) + iv(x,y)$$

is the given function. Such a criterion is given in the following theorem.
Theorem VII. In a given finite region $S$ let $u$ and $v$ be two real single-valued functions of the real variable $x, y$. The necessary and sufficient condition that the complex function

$$w = u + iv$$

is holomorphic in $S$ is that the partial derivatives of $u$ and $v$ of the first order exist and are continuous and moreover satisfy the following partial differential equation:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad (1)$$

Proof: These differential equations are known as the Cauchy-Riemann differential equations. To show that these equations present a necessary condition we proceed as follows. Since the function $w$ is holomorphic in $S$, it has a derivative with respect to $z$. As we have seen, the existence of this derivative involves the condition that the ratio $\frac{\Delta w}{\Delta z}$ shall have the same limiting value as $\Delta z$ approaches zero in any direction whatsoever. Consequently, the same limiting value is obtained if $\Delta z$ is permitted to approach zero through real values or through purely imaginary values. The increment $\Delta z = \Delta x + i \Delta y$ becomes in the first case $\Delta x$ and in the second case $i \Delta y$. We may therefore write

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta x} = \lim_{\Delta z \to 0} \frac{1}{i} \frac{\Delta w}{\Delta y}. \quad (2)$$

Since the first limit exists by hypothesis, the second and third limits must also exist. We have then

$$\frac{dw}{dz} = \frac{dw}{dx} = \frac{1}{i} \frac{dw}{dy}. \quad (3)$$
However, we have
\[ \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \]
\[ = \frac{\partial u}{\partial x} - i \frac{\partial v}{\partial x} \]  
\[ \frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \]
\[ = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \]  
(4)

where \( w = u(x,y) + iv(x,y) \).

Substituting these values in (3), we obtain
\[ \frac{dw}{dz} = \frac{\partial u}{\partial x} - \frac{1}{i} \frac{\partial v}{\partial x} = \frac{1}{i} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right) \]
\[ = \frac{1}{i} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \]
\[ = \left( \frac{1}{i} \cdot \frac{1}{i} \right) \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \]
\[ = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \]  
(6)

Equating the real and imaginary parts in this equation, we have
\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \]  
(7)

By Theorem IV, the derivative \( \frac{dw}{dz} \) is continuous. The continuity of the partial derivatives in (7) follows therefore from equations (3), (4), and (5) and from the definition of continuity.

We may now show the conditions of the theorem to be sufficient as follows. Since \( f(z) \) is continuous in \( z \) along \( C \) and
and that both \( u(x,y), v(x,y) \) are continuous in \( x,y \) together along the same curve, and moreover, as \( \Delta z \to 0 \) we have \( \Delta x \to 0, \Delta y \to 0 \). Hence, upon passing to the limit, we obtain

\[
\int_C f(z) \, dz = \int_C u \, dx - v \, dy + i \int_C v \, dx - u \, dy, \tag{8}
\]

where \( C \) may be regarded as any path of integration within the given region \( S \). As \( u, v \) are continuous, both of the integrals

\[
\int_C u \, dx - v \, dy, \quad \int_C v \, dx - u \, dy \tag{9}
\]

exist. By hypothesis, the equations (1) are satisfied by \( u, v \). In order for both integrals in (9) to be zero let us assume the proof of the following theorem to be true.

Theorem. In a given finite region \( S \) let \( C \) be the complete boundary of any portion of the plane such that \( C \) lies within \( S \) and incloses only points of \( S \). If in the given region \( P(x,y), Q(x,y) \) are continuous real functions of \( x \) and \( y \) together, having the continuous partial derivatives \( \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y} \), then the necessary and sufficient condition that the integral

\[
\int_C P \, dx - Q \, dy
\]

vanishes for every such curve \( C \) is that

\[
\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}
\]

for all points of \( S \) (7;56,57).

With this assumption, both of the integrals in (9) are zero, and we have from (8)

\[
\int_C f(z) \, dz = 0
\]
for every path of integration \( C \) forming a complete boundary of any portion of \( S \) such that \( C \) lies entirely within \( S \) and incloses only points of \( S \). Consequently, by Theorem \( V \), \( f(z) \) is holomorphic in the given region \( S \) as the theorem requires.

From equations (3), (4), and (5), we have

\[
\frac{dw}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y},
\]

This relation affords a convenient method of computing the derivatives of \( w \), when \( w = f(z) \) is expressed in terms of \( x \) and \( y \).

**Demoivre's Theorem.**—If \( n \) is any positive whole number,

\[
(cos \psi + i \sin \psi)^n = \cos n \psi + i \sin n \psi. \tag{1}
\]

Proof: This relation is evidently true when \( n = 1 \), and when \( n = 2 \) it follows from formula (1), Article 3, with \( \psi = \psi \). To proceed by mathematical induction, suppose that our relation has been established for the values 1, 2, ..., \( m \) of \( n \). We can then prove that it holds also for the next value \( m + 1 \) of \( n \). For, by hypothesis, we have

\[
(cos \psi + i \sin \psi)^m = \cos m \psi + i \sin m \psi.
\]

Multiply each member by \( \cos \psi + i \sin \psi \), and for the product on the right substitute its value from (1), Article 3, with \( \psi = m \psi \). Thus

\[
(cos \psi + i \sin \psi)^m \cdot (cos \psi + i \sin \psi) = (cos \psi + i \sin \psi)(cos m \psi + i \sin m \psi),
\]

\[
= \cos (\psi + m \psi) + i \sin (\psi + m \psi),
\]

which proves (1) when \( n = m + 1 \). Hence the induction is
Laplace's Differential Equation. — We have the following theorem.

Theorem IX. In a given finite region S, let the complex function

\[ f(z) = u + iv \]

be holomorphic; then the functions \( u(x,y) \), \( v(x,y) \) satisfy the partial differential equation

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \]  

(1)

Proof: Since \( f(z) \) is holomorphic in \( S \), the derivative \( f'(z) \) and also the higher derivatives exist and are holomorphic in the same region. It follows that the partial derivatives of \( u,v \) with respect to \( x \) and \( y \) exist and are continuous. This statement holds not only for the partial derivatives of the first order but likewise for those of the second and higher orders. From the Cauchy-Riemann differential equations

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \]

(2)

differentiating we get

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x}. \]

(3)

As the partial derivatives of the second order are continuous in \( x,y \), together, we have (5:426)

\[ \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}. \]

(4)
From (4) and by addition of the equations (3), we have

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad (5) \]

In a similar manner we shall show that

\[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \quad (6) \]

From the Cauchy-Riemann differential equations,

\[ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \quad (7) \]

we obtain by differentiating

\[ \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}. \quad (8) \]

Applying the same step in getting (4)

\[ \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}. \quad (9) \]

And by addition of the equations in (8), we have

\[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \]
CHAPTER III

MAPPING IN GENERAL WITH APPLICATIONS TO CERTAIN FUNCTIONS

Mapping. — Real functions of real variables, \( y = f(x) \) can be exhibited graphically by plotting corresponding values of \( x \) and \( y \) as rectangular coordinates of points in the XY-plane. The function \( f(x) \) maps each point on the x-axis into a point in the plane at a directed distance \( y \) above or below that point. The result of mapping all points of the x-axis is a curve, the graph of the function.

When the variables are complex, the graphical representation of functions is more complicated, for if \( w = f(z) \), each of the complex variables \( w \) and \( z \) are represented geometrically by points in the complex plane. For convenience, the representation of the \( z \)-points are in one plane called the Z-plane, and the \( w \)-points are in another plane called the W-plane. These two planes have a relation to each other similar to that which the two coordinate axes have in the consideration of functions of a real variable. As the point \( P \) traces a curve in the Z-plane the corresponding point \( Q \) will trace a curve in the W-plane. The relationship between the two curves may be expressed by saying the Z-plane is mapped upon the W-plane (Figures 8 and 9, Page 32).

The correspondence between points in the two planes may be defined as a MAPPING or a TRANSFORMATION of points in the
Figure 8.

Figure 9.
Z-plane into points in the W-plane by the function \( W = f(z) \).

Corresponding points are called images of each other. The word image is also applied to a curve or a region in one plane corresponding to a curve or region in the other.

The mapping of corresponding curves and regions in the two planes usually gives more information about the function than the mapping of individual points.

Even though two separate planes are used to represent \( W \) and \( Z \), it is often convenient to think of the mapping as effected in one plane thus permitting the use of such graphic terms as translation and rotation.

In mapping we may speak of the mapping of one plane upon the other rather than of the mapping of some particular configuration from the one plane upon the other. If \( f(z) \) is multiple-valued, then to each point in the \( Z \)-plane there correspond in general several distinct points in the \( W \)-plane. In such cases it is often convenient to map the whole of one plane upon a portion of the other.

We shall now do mapping with application to certain functions, these functions being, \( w = z^2 \), \( w = z^n \) (\( n \) as a positive integer) and \( w = \log z \).

**The Function \( w = z^2 \).** Let us consider the function

\[
    W = Z^2.
\]

We have then

\[
    W = u + iv = (x + iy)^2 = x^2y^2 + 2ixy. \quad (1)
\]
Equating the real and imaginary parts, we get
\[ u = x^2 - y^2, \quad v = 2xy. \quad (2) \]
We may also write
\[ z = \rho (\cos \theta + i \sin \theta) \quad (3) \]
\[ w = \rho' (\cos \theta' + i \sin \theta') = \rho^2 (\cos 2\theta + i \sin 2\theta). \]
Hence, we have
\[ \rho' = \rho^2, \quad \theta' = 2\theta \quad \text{(defined in Chapter III)}. \]
It appears that the amplitude of \( w \) is twice the amplitude of \( z \), so that if \( z \) describes an arc subtending an angle \( \theta \) at the origin, \( w \) describes an arc which subtends an angle \( 2\theta \) at its origin. Hence, half of the \( Z \)-plane maps in the whole of the \( W \)-plane and on the other hand a half of the \( W \)-plane maps into the first quadrant of the \( Z \)-plane.

From above we know what part of the \( W \)-plane is imaged on the \( Z \)-plane. Now let us see what is imaged.

To map from the \( W \)-plane upon the \( Z \)-plane, suppose we put \( u = c \), a constant. We obtain a hyperbola given by the equation
\[ x^2 - y^2 = c. \quad (4) \]
Equation (4) may take the form
\[ \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1, \]
or
\[ \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \quad (5) \]
where \( a^2 = b^2 = c \) with center \((h, k) = 0\). When \( u = c > 0 \), the intercepts are
\[ xx' = t\sqrt{c} \quad \text{(real)} \]
\[ yy' = ti\sqrt{c} \quad \text{(imaginary)}. \]
If \( f^2 = a^2 \neq b^2 \) or \( f^2 = c \neq 0 \), then the foci are \( (\pm \sqrt{2c}, 0) \).
The length of the latus rectum is \( 2a \). When \( a = c < 0 \),
the intercepts are
\[
xx' = f \sqrt{a} \quad \text{(imaginary)}
\]
\[
yy' = f \sqrt{a}.
\]
The foci are \( (0, \pm \sqrt{2c}) \) and the length of the latus rectum
\[
2a = \sqrt{b}.
\]

If in one of the typical forms of the equation of a hyperbola we replace the constant term by zero, the locus of the new equation is a pair of lines which are called asymptotes of the hyperbola.

Thus the asymptotes of the hyperbola
\[
x^2 - y^2 = c
\]
are the lines
\[
x^2 - y^2 = 0, \text{ or } x^2 - y^2 = 0.
\]
These may be written
\[
y = \pm x \quad \text{and } y = \mp x.
\]
Therefore the hyperbolas will have the lines \( y = \pm x \) as asymptotes according as \( c \) is positive or negative.

For \( v = c' \), a positive constant, we obtain in the \( \mathbb{C} \)-plane a system of hyperbolas orthogonal to the first systems given by the equation
\[
2xy = c',
\]
or
\[
xy = \frac{c'}{2}. \quad (6)
\]
The plotting of these hyperbolas with the axes as asymptotes are drawn in the first and third quadrant but we will only see the hyperbolas in the first quadrant.

We shall now plot a few curves belonging to equations (4) and (6), respectively.

1. The case \( u = c = t \).
   
   intercepts: \( xx' = \frac{t}{1}, \quad yy' = \frac{t}{1} \)
   
   foci: \( (t, 0), \quad (0, \pm t) \)
   
   length of latus rectum is 2.

2. The case \( u = c = t \).

   intercepts: \( xx' = \frac{t}{\sqrt{2}}, \quad \frac{t}{1} \)
   
   foci: \( (t, 0), \quad (0, \pm t) \)
   
   length of latus rectum is 2.8.

3. The case \( u = c = t \).

   intercepts: \( xx' = \frac{t}{\sqrt{3}}, \quad \frac{t}{1} \sqrt{3} \)
   
   foci: \( (t, 0), \quad (0, \pm t) \)
   
   length of latus rectum is 3.5.

4. The case \( u = c = t \).

   intercepts: \( xx' = \frac{t}{2}, \quad \frac{t}{2} \)
   
   foci: \( (t, 0), \quad (0, \pm t) \)
   
   length of latus rectum is 4.
5. The case \( v = c' = 1 \).

\[
\begin{array}{c|c|c|c}
 x & y & x & y \\
 1 & \frac{1}{2} & \frac{1}{2} & 1 \\
 2 & \frac{1}{4} & \frac{1}{4} & 2 \\
 3 & \frac{1}{6} & \frac{1}{6} & 3 \\
 4 & \frac{1}{8} & \frac{1}{8} & 4 \\
\end{array}
\]

6. The case \( v = c' = 2 \).

\[
\begin{array}{c|c|c|c}
 x & y & x & y \\
 1 & 1 & 1 & 1 \\
 2 & \frac{1}{2} & \frac{1}{2} & 2 \\
 3 & \frac{1}{3} & \frac{1}{3} & 3 \\
 4 & \frac{1}{4} & \frac{1}{4} & 4 \\
\end{array}
\]

7. The case \( v = c' = 3 \).

\[
\begin{array}{c|c|c|c}
 x & y & x & y \\
 1 & \frac{1}{2} & \frac{1}{2} & 1 \\
 2 & \frac{3}{4} & \frac{3}{4} & 2 \\
 3 & \frac{1}{6} & \frac{1}{6} & 3 \\
 4 & \frac{3}{8} & \frac{3}{8} & 4 \\
\end{array}
\]
8. The case \( v = c' = 4 \).

The mapping graphically will look like Figures 10 and 11.

The Function \( w = z^n \): We have just completed a discussion on the special case of this general function \( w = z^n \), where \( n \) is a positive integer.

In the discussion of the function \( w = z^2 \), attention was called to the fact that half of the \( Z \)-plane maps into the whole of the \( W \)-plane. Let us now consider the general case, where \( w = z^n \). We have

\[
\begin{align*}
z &= \rho (\cos \theta + i \sin \theta) \\
w &= u + i v = \rho (\cos \theta' + i \sin \theta') = \rho^n (\cos n\theta + i \sin n\theta).
\end{align*}
\]

We have upon equating the real and imaginary parts, squaring and adding,

\[
\begin{align*}
u &= \rho^n \cos n\theta, \\
v &= \rho^n \sin n\theta, \\
u^2 + v^2 &= \rho^{2n}.
\end{align*}
\]
Figure 10.

Figure 11.
which is the equation of a circle with radius $\rho^n$.

From the relations (1), (2),

$$\rho' = \rho^n, \quad \theta' = n\theta.$$ 

The amplitude of $w$ is $n$ times the amplitude of $z$. From the relation between $\theta$ and $\theta'$, it will be seen that $\left(\frac{1}{n}\right)^{th}$ of the circle in the $Z$-plane maps into the whole of the circle in the $W$-plane; consequently, a sector bounded by any two half rays drawn from the origin making an angle $\frac{2\pi}{n}$ radians with each other maps into the whole of the $W$-plane. In order for the above to be true $\theta' = 2\pi$ radians, therefore $\theta = \frac{2\pi}{n}$ radians. The values of $\theta'$ corresponding to the chief amplitude of $w$ belongs to the interval

$$-\pi < \theta < \pi.$$ 

The sector bounded by $OR_1$, $OR_2$, making respectively the angles $\frac{\pi}{n}$, $-\frac{\pi}{n}$ with the positive $x$-axis, maps in a continuous single-valued manner upon the entire $W$-plane. The lower bank of the line $OR_1$ goes over into the upper bank of the negative axis of reals of the $W$-plane. The upper bank of the line $OR_2$ maps into the lower bank of the negative axis of reals of the $W$-plane as shown in Figures 12 and 13, Page 41.

**The Function $w = \log z$.** - In real variables the logarithm is frequently defined as the inverse function of $e^z$. We may use this property in defining logarithm of a complex number. To determine the inverse of the exponential function let us consider the relation

$$e^w = z.$$ 

(1)
We have, as elsewhere,
\[ w = u + iv \]
\[ z = x + iy. \] (2)

Substituting in equation (1) we get
\[ e^{u+iv} = x + iy, \]
\[ e^u e^{iv} = x + iy. \] (3)

But \( e^{iv} = (\cos v + i \sin v) \) and replacing this form in (3),
we have
\[ e^u (\cos v + i \sin v) = x + iy. \] (4)

Equating the real and imaginary parts, we then have
\[ x = e^u \cos v \]
\[ y = e^u \sin v. \] (5)

We may assign values to \( x \) and \( y \). For \( x = 0 \) is the map of
certain lines parallel to the \( u \)-axis. Any line parallel to the
\( u \)-axis maps into a half-ray in the \( Z \)-plane proceeding from the
origin. This may be shown as follows. Eliminating \( u \) from
equations (5) by multiplying the first of these equations by \( \sin v \)
and the second by \( \cos v \) and subtracting we have
\[ x \sin v = e^u \sin v \cos v \]
\[ y \cos v = e^u \sin v \cos v \]
\[ x \sin v - y \cos v = 0 \]

For constant values of \( v \), this equation gives straight lines of
the form
\[ y = mx, \]
where \( m = \tan v \) in the following manner. Dividing equation (6)
by \( \cos v \), we have
\[ \frac{x \sin v}{\cos v} = y \frac{\cos v}{\cos v} \]

from trigonometry \( \frac{\sin v}{\cos v} = \tan v \),
\[ x \tan v = y \]
\[ y = mx. \]

Since \( e^u \) is positive for all finite values of \( u \), it follows from (5) that any line \( v = c \) maps into a half-ray from the origin taken along the line \( y = mx \); the portion of this half-ray interior to the unit circle corresponds to negative values of \( u \), while the portion exterior to the unit circle corresponds to positive values of \( u \). If successive values of \( v \) differ by equal amounts, then the corresponding half-rays in the \( \mathbb{C} \)-plane will make equal angles with each other.

In a similar manner, for \( y = 0 \) in (5), is the map of a line parallel to the \( v \)-axis which is easily obtained as follows. Eliminating \( v \) from the equations
\[ x = e^u \cos v, \quad y = e^u \sin v \]
by squaring and adding, we have
\[ x^2 + y^2 = e^{2u} (\cos^2 v + \sin^2 v) = e^{2u}. \]
For any constant value of \( u \), we have then a circle in the \( \mathbb{C} \)-plane about the origin as a center.

For
\[ u = 0, \]
we have
\[ x^2 + y^2 = 1; \]
that is, the \( v \)-axis maps into the unit circle about the origin in the \( \mathbb{C} \)-plane. For \( u = e > 0 \), the map in the \( \mathbb{C} \)-plane is a
circle exterior to the unit circle; and for \( u = c < 0 \), the map is a circle lying within the unit circle. If \( a = u < 0 \) then

\[
x^2 + y^2 = \frac{1}{e^{2a}}
\]

and hence, the radius will be less than 1 and therefore will lie within the unit circle.

The mapping of this function graphically is as follows. The unit circle is indicated by broken lines. For \( u = c > 0 \), the circles lie exterior to the unit circle. For \( u = c < 0 \), interior to the unit circle (Figures 14 and 15, Page 45).
Figure 14.

Figure 15.
CHAPTER IV
APPLICATIONS OF MAPPING BY MEANS OF SOME DEFINITE FUNCTIONS

Discuss the conjugate functions determined by the relation
\[ w^2 = z / l. \]

Plot the projections upon the XY-plane of the lines of level and lines of slope.

Solution:
\[ w^2 = z / l \]
where
\[ w = u / iv \]
\[ z = x / iy. \]

We have then
\[ w^2 = (u / iv)^2 = x / l / iy. \]

Equating the real and imaginary parts, we get
\[ u^2 - v^2 = x / l, \quad 2uv = y. \tag{1} \]

Solving the equations (1) simultaneously we obtain values for
\[ u = u(x, y) \text{ and } v = v(x, y) \]
\[ \begin{aligned}
    u^2 - v^2 &= x / l \\
    2uv &= y
\end{aligned} \]
\[ u = \frac{-y}{2v}. \tag{2} \]

Substituting the value for \( u \) in the equation
\[ u^2 - v^2 = x / l, \]
we get
\[ 4 \frac{u^2}{v^2} - v^2 = x / l \]
\[ y^2 - 4v^4 = 4v^2(x / l) \]
\[ 4v^4 / 4(x / l)v^2 - y^2 = 0. \]
therefore

\[ v = \frac{-4(x + 1) \sqrt[4]{(x + 1)^2 + y^2}}{8} \]

Replacing \( v \) in (2) by its value in (3), we obtain

\[ u = \frac{y}{\sqrt{2x - (x + 1) - (x + 1)^2 + y^2}} \]  

The given equation may be written in polar coordinate form. If we let \( z / 1 = z' = \rho' (\cos \theta' + i \sin \theta') \) and \( w = \rho (\cos \theta + i \sin \theta) \), then

\[ w^2 = \rho^2 (\cos 2\theta + i \sin 2\theta) = \rho (\cos \theta' + i \sin \theta'). \]

Hence, we have

\[ \rho^2 = \rho', \quad 2\theta = \theta'. \]

From the relation between \( \theta \) and \( \theta' \), it will be seen that a half of the \( W \)-plane maps into the whole of the \( Z \)-plane.

To map from the \( W \)-plane upon the \( Z \)-plane, suppose we put \( u = c \), a constant.

From (4)

\[ c = \frac{y}{2 \sqrt{-(x + 1) + \sqrt{(x + 1)^2 + y^2}}} \]

\[ y = 2c \sqrt{-(x + 1) + \sqrt{(x + 1)^2 + y^2}} \]
\[
\begin{align*}
\frac{48}{y^2} &= 2c^2 \left[ -(x - 1) + \sqrt{(x - 1)^2 + y^2} \right] \\
\frac{y^2}{x^2} - \frac{2c^2}{2c^2} &= 4c^4(x + 1)^2 - 4c^4y^2 \\
\frac{y^4}{4c^4x^2} - \frac{4c^4}{4c^4} + \frac{4c^4xy^2}{8c^4x} - 4c^4x^2 &= 0 \\
\frac{y^2}{4c^2x} - \frac{4c^2 - 4c^4}{4c^4} &= 0 \\
y^2 &= -4c^2x + 4c^4 - 4c^2 \\
y^2 &= -4c^2(x - c^2 + 1). \\
\tag{5}
\end{align*}
\]

We obtain a parabola given by equation (5) with vertex at \((h, k)\) with axis parallel to the \(x\)-axis when equation (5) takes the form

\[
(y - k)^2 = 2p(x - h),
\]

\[p = -2c^2, \; k = 0, \; \text{and} \; h = c^2 - 1,
\]

having as its foci \((h - p, k)\) and latus rectum \(2p\).

When we put \(\mu = c\), a constant, the \(w\)-plane consists of lines parallel to the \(y\)-axis. In assigning \(c\) the values \(\pm 1, \ldots, \pm n\), the assigned value of \(c\) whether positive or negative maps into one curve in the \(z\)-plane. If we put \(\mu = c = 0\), the \(v\)-axis maps into the \(x\)-axis in the \(z\)-plane. For \(v = c'\), a constant, we have from (3)

\[
c' = \sqrt{-(x + 1) + \sqrt{(x + 1)^2 + y^2}} \\
2c'^2 = -(x + 1) + \sqrt{(x + 1)^2 + y^2} \\
(2c'^2 + x + 1)^2 = (x + 1)^2 + y^2 \\
4c'^2 - x^2 + 1 + 4c'^2 - 2x = x^2 + 2x + 1 + y^2 \\
y^2 = 4c'^2x + 4c'^4 + 4c'^2 \\
y^2 = 4c'^2(x + c'^2 + 1).
\]
When \( v = c' \), a constant, the \( W \)-plane consists of lines parallel to the \( u \)-axis and maps upon the \( Z \)-plane parabolas with vertex at \((h', k')\) with axis parallel to the \( x \)-axis. Equation (6) may take the form

\[
(y - k')^2 = 2p'(x - h'),
\]

where \( p' = 2c'^2 \), \( k' = 0 \), and \( h = -1 - c'^2 \)

having as its foci \((h' + \frac{p'}{2}, k')\) and latus rectum \(2p'\).

Likewise in (5), the assigned value of \( c \) whether positive or negative maps into one curve in the \( Z \)-plane. Also, when \( v = c' = 0 \), the \( u \)-axis maps into the \( x \)-axis in the \( Z \)-plane.

In the \( Z \)-plane we shall denote \( u = c = 0 \) by this line \( \ldots \ldots \ldots \) and \( v = c' = 0 \) denote by this line \( \ldots \ldots \ldots \) . The line \( \ldots \ldots \ldots \) on the upper bank of the \( x \)-axis does not indicate the upper half of the \( Z \)-plane and the line \( \ldots \ldots \ldots \) on the lower bank of the \( x \)-axis does not indicate the lower half of the \( Z \)-plane but rather the \( x \)-axis itself. For the line \( u = c = 0 \) and the line \( v = c' = 0 \) map upon the same axis.

We shall now plot a few curves belonging to equations (5) and (6), respectively.

1. The case \( u = c = 1 \).
   - vertex \((0, 0)\), foci \((-1, 0)\), and latus rectum 4.

2. The case \( u = c = 2 \).
   - vertex \((3, 0)\), foci \((-1, 0)\), and latus rectum 16.

3. The case \( u = c = 3 \).
   - vertex \((8, 0)\), foci \((-1, 0)\), and latus rectum 36.

4. The case \( u = c = 4 \).
The vertex is at \((13,0)\), the foci are at \((-1,0)\), and the latus rectum is 64.

5. The case \(v = c' = 1\).
   The vertex is at \((-2,0)\), the foci are at \((-1,0)\), and the latus rectum is 4.

6. The case \(v = c' = 2\).
   The vertex is at \((-5,0)\), the foci are at \((-1,0)\), and the latus rectum is 16.

7. The case \(v = c' = 3\).
   The vertex is at \((-10,0)\), the foci are at \((-1,0)\), and the latus rectum is 36.

8. The case \(v = c' = 4\).
   The vertex is at \((-17,0)\), the foci are at \((-1,0)\), and the latus rectum is 64.

The mapping graphically of this function is shown in Figures 16 and 17, Page 51.

Discuss the mapping upon the \(W\)-plane of a system of concentric circles about the origin in the \(Z\)-plane, by means of the relation

\[ w = \frac{3z^3}{5}. \]

Solution: \( w = \frac{3z^3}{5}. \) \hspace{1cm} (1)

If we let \( w = 5 = w' \) and writing equation (1) in polar coordinate form, we get

\[ z = \rho(\cos \theta + i \sin \theta), \]

\[ w' = \rho'(\cos \theta' + i \sin \theta') = 3\rho^3(\cos 3\theta + i \sin 3\theta). \]

Hence, we have

\[ \rho' = 3\rho^3, \quad \theta' = 3\theta. \]

From the relation between \( \theta \) and \( \theta' \), it will be seen that a third of the \( Z\)-plane maps into the whole of the \( W\)-plane.

We may also write

\[ w = u + iv = 3\rho^3 (\cos 3\theta + i \sin 3\theta) + 5. \] \hspace{1cm} (2)
Figure 16.

Figure 17.
Equating the real and imaginary parts, we obtain

\[ u = 3 \rho^3 \cos 3\theta + 5, \quad v = 3 \rho^3 \sin 3\theta, \]

or

\[ \frac{u - 5}{3\rho^3} = \cos 3\theta, \quad \frac{v}{3\rho^3} = \sin 3\theta. \quad (3) \]

Squaring equations in (3) and adding, we have

\[ \frac{(u - 5)^2}{9\rho^6} + \frac{v^2}{9\rho^6} = 1, \]

or

\[ (u - 5)^2 + v^2 = 9\rho^6. \]

If \( \rho = c \), \( c \) is an arbitrary constant, we have

\[ (u - 5)^2 + v^2 = 9c^6. \quad (4) \]

From the above assumption we would have in the \( Z \)-plane concentric circles about the origin (hypothesis) and from equation (4) will map upon the \( W \)-plane circles whose center is \((5,0)\), radius \( 3c^3 \). The circles in the \( W \)-plane are very much larger than the circles in the \( Z \)-plane (Figures 18 and 19, Page 53).

We shall now verify the above statement.

1. When \( \rho = 1 \) (\( Z \)-plane), \( c = 1 \) and radius is 3 in the \( W \)-plane.
2. When \( \rho = 2 \) (\( Z \)-plane), \( c = 2 \) and radius is 24 in the \( W \)-plane.
3. When \( \rho = 3 \) (\( Z \)-plane), \( c = 3 \) and radius is 81 in the \( W \)-plane.

Discuss the function

\[ w = \log \frac{(z - 1)^3}{z^2}. \]
and point out possible applications as suggested by the map in the $Z$-plane of the lines $u = c$, $v = c$. Locate the points of equilibrium, if such exist.

Solution: $w = \log \frac{(z - 1)^3}{z^2}$.

If we put

$$z - 1 = \rho e^{i\theta}, \quad z = \rho e^{i\theta}, \quad w = u + iv,$$

then

$$w = u + iv = \log \frac{\rho^3 e^{3i\theta}}{\rho^2 e^{2i\theta}}$$

$$= \log \frac{\rho^3}{\rho^2} + i(3\theta - 2\theta)$$

$$= \log \frac{\rho^3}{\rho^2} + i(3\theta - 2\theta).$$

Hence, we obtain

$$u = \log \frac{\rho^3}{\rho^2}, \quad v = 3\theta - 2\theta.$$  

For $u = c$, we have

$$c = \log \frac{\rho^3}{\rho^2},$$

or

$$\rho_1^2 = \frac{\rho^3}{\rho^2} = K\rho_1^3, \quad K > 0$$

$$\rho_1 = \sqrt{K}\rho_1^3$$ \hspace{1cm} (1)

For the orthogonal system, we have

$$c = 3\theta - 2\theta.$$ \hspace{1cm} (2)

To plot any one of the system of curves represented by (1), give $K$ an assigned value and give to $\rho$ any convenient succession of values. Assigning $K$ the value of $\frac{1}{4}$ and $\rho_1$ the values $1, 2, 3, \ldots$, we have for
\[ \rho_1 = 1, \quad \rho_2 = \frac{1}{2} \]
\[ \rho_1 = 2, \quad \rho_2 = 1.4 \]
\[ \rho_1 = 3, \quad \rho_2 = 1\frac{1}{3} \]

With \( z = 1 \) as a center and the assumed values of \( \rho \) as radii, draw circles. Likewise, with \( z = 0 \) as a center and the computed values of \( \rho \) as radii draw circles. The intersection of corresponding circles give points on the required curve.

To plot a curve belonging to the system given by (2), give to \( \theta \) any assigned value and from the points \( z = 1, z = 0 \), and draw lines making angles \( \theta_1 \) and \( \theta_2 = \frac{3\theta}{2} - \alpha \), respectively, with the axis of reals. Assigning \( \theta \) the value of 0 and \( \theta \), a succession of values \( \pm \frac{\pi}{4}, \pm \frac{2\pi}{5}, \pm \frac{7}{6} \), \( \ldots \), expressed in radians, we have for

\[ \theta_1 = \frac{\pi}{4} = 45^\circ, \quad \theta_2 = 39^\circ \]
\[ \theta_1 = \frac{\pi}{3} = 30^\circ, \quad \theta_2 = 26^\circ \]
\[ \theta_1 = \frac{\pi}{2} = 45^\circ, \quad \theta_2 = 17^\circ \]

The intersection of corresponding lines gives points on the required curve. The general form of the two systems of curves is shown in Figure 21, Page 56.

To determine the double points of the lines of force, that is, the points of equilibrium, we have

\[ u = \log \frac{\rho^3}{\xi^2} = \log \sqrt{(x - 1)^2 + y^2} \sqrt{\frac{(x - 1)^2 + y^2}{x^2 + y^2}} \]
\[ u = \log \sqrt{(x - 1)^2 + y^2} \frac{\rho^2}{\xi^2} - \log (x^2 + y^2) \]

The double points are given by putting partial derivatives of \( u \)
with respect to \( x \) and \( y \) equal to zero and solving the two resulting equations for \( x \) and \( y \). We have then to solve the equations

\[
\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \log \frac{(x - 1)^2}{\sqrt{y^2 + 2y}} - \frac{\partial}{\partial x} \log \frac{x^2 + y^2}{y^2}
\]

\[
= \frac{3}{2} \frac{(x - 1)^2}{x^2 + y^2} - \frac{2(x - 1) - \frac{3}{2} \log (x^2 + y^2)}{x^2 + y^2}
\]

\[
= \frac{3(x - 1)}{(x - 1)^2 + y^2} - \frac{2(x - 1)}{x^2 + y^2} = 0,
\]

\[
\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \log \frac{(x - 1)^2}{\sqrt{y^2 + 2y}} - \frac{\partial}{\partial y} \log \frac{x^2 + y^2}{y^2}
\]

\[
= \frac{3y}{(x - 1)^2 + y^2} - \frac{2y}{x^2 + y^2} = 0.
\]

The two equations set equal to zero are satisfied simultaneously by the values \( y = 0, x = -2 \). These values are therefore the coordinates of the point \( C \) of equilibrium. To determine which one of the lines \( u = c \) maps into the particular curve having a double point at \((-2,0)\), we substitute the values \( x = -2, y = 0 \) in (1) and determine the corresponding value of \( c \). This substitution gives

\[
c = \log \frac{x^3}{y^2} = \log \frac{(x - 1)^2 + y^2}{\sqrt{(x - 1)^2 + y^2}}
\]

\[
= \frac{\log \left[ \log \left[ \left( (-2)^2 \right) \right] \right]}{(-2)^2}
\]
\[
\log \frac{27}{4}
\]

\[= \log 6.25,\]

that is, the potential function has at each point of this curve the value \(\log 6.25\).
BIBLIOGRAPHY


