8-1-1980

Algebraic categories

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ALGEBRAIC CATEGORIES

A THESIS SUBMITTED TO THE FACULTY OF
ATLANTA UNIVERSITY IN PARTIAL FULFILLMENT ON
THE REQUIREMENTS FOR THE MASTER OF SCIENCE DEGREE

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ATLANTA, GEORGIA

AUGUST 1980
Chapter I

Introduction
Abstract
Mathematics

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July 1980

In 1945 Samuel Eilenburg and Saunders MacLane Published a mathematical paper titled the "General Theory of Natural Equivalences". This paper introduced the concept of categories with the following definition:

A category is formed with a class, $\mathcal{C}$, of objects $A, B, C, \ldots$ where there is a set of mappings (morphisms) between any distinct pair of those objects of $\mathcal{C}$ such that there are functions $f: A \rightarrow B$ in the set of $\text{hom}(A, B)$ together with any triple $(A, B, C)$ of $\mathcal{C}$ a composition map of morphisms:

$\text{hom}(B, C) \times \text{hom}(A, B) \rightarrow \text{hom}(A, C)$ (or with functions $f: A \rightarrow B$, $g: B \rightarrow C$ and $h: A \rightarrow C$ such that $g \circ f = h$) such that the following conditions hold:

1. Associativity of compositions $(h \circ g) \circ f = h \circ (g \circ h)$
2. Identity of Composition

For any object $B$, there is the identity $1_B : B \rightarrow B$ such that for $f: A \rightarrow B$ and $g: B \rightarrow C$ then $1_B \circ f = f$ and $g \circ 1_B = g$. 
The essence of this definition provided mathematicians a way to map "large" mathematical structures into other large mathematical structures which provides mathematicians a way to compare those structures. Over the thirty five year period the area of category has grown extensively.

The purpose of this thesis is to examine algebraic categories - a specific area of category theory. This has been done by first studying the basic properties of all categories (duality, equivalence, universality, morphisms and functors), second by definition and citing specific examples and third by giving theorems and proofs of properties of algebraic categories and functors.

In this study it was determined that algebraic categories are formed with a concrete category \( \mathcal{A} \) and a forgetful functor \( U: \mathcal{A} \to \text{Set} \), where the following properties hold:

1. \( \mathcal{A} \) has coequalizers
2. \( U \) has a left adjoint
3. \( U \) preserves and reflects epimorphisms

Theorems were found that prove an algebraic category \( (\mathcal{A}, U) \) preserves equalizers, products, pullbacks and limits. Other theorems show that if category \( \mathcal{A} \) has coequalizers and if \( U: \mathcal{A} \to \text{set} \) then \( (\mathcal{A}, U) \) is an algebraic category. It was learned that algebraic functors preserve and reflect monomorphisms and isomorphism. Finally, it was proved that a mapping of two algebraic categories is algebraic.
The main sources of this thesis are *Category Theory* by Horst Herrlich and George E. Strecker, *Categories* by Horst Schubert, as paper "on Algebraic Categories" by George Georges and Dori Popescu and *Lecture Notes in Mathematics: Categories of Algebraic Structures* by Mario Petrich.
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Bibliography
Historical Prospectus

In the early nineteen forties Samuel Eilenberg and Saunders MacLane, two algebraic topologist, did some work on a topological problem which lead to the formation of category theory. Note that "Topology is the study of those properties of spaces that are invariant under homeomorphic transformations (one to one and continuous both ways)."¹ "Algebraic topology attempts to construct an algebraic model of topological properties."² This field produced the theory of homology. "This arose from one problem of orientation. It had been found that problems arising from line surface, and volume integrals required a convention under which the positive or negative direction of a curve or surface, or the corresponding entity in a higher dimension, could be specified. For a topological space X of m + 1 dimensions, a theory of m-cycles was evolved in which given the orientations the law of addition and substraction of cycles was apparent. It could also be seen that the set of m-cycles under the operation of addition, formed a group. Any two cycles whose difference formed a cycle that bounded the space are said to be homologous, and to belong to the same homology class. It can be shown that the homology classes are groups under the operation of cycle addition. The usefulness of this concept lies in the fact that

² Ibid., P. 121 and 122.
homeomorphic spaces have isomorphic homology groups. Thus, by studying this group structure, homeomorphic correspondences between spaces can be found. These homeomorphic correspondences between topological spaces were characterized by Eilenberg and MacLane using the concepts of categories. Eilenberg explains as follows:

"Let X and Y be two topological spaces and denote by (X, Y) the set of all continuous mappings from X into Y. If we have two continuous maps, one from a topological space X into a topological space Y and one from Y into a third topological space \( \mathbb{Z} \), then we have a composite map from X into \( \mathbb{Z} \). This composition of maps is associative. Also, for every topological space X we have the identity map from X onto itself, which composed with any map into X or from X gives us that map again. Thus we obtain an algebraic structure called a category; the topological spaces are called the objects of this category and the continuous maps its morphisms."

Eilenberg and MacLane published the foundations for their work on categories in 1942 in a paper titled "Group Extensions and Homology". In 1945 these algebraic topologist published a paper that established the fundamental definitions theorems involving categories and functors of abstract objects. Since 1945 category theory has expanded and is used in many areas of mathematics.

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Objectives

The historical prospectus states that the concepts of categories have implications for several areas of mathematics. This paper will focus on one of those areas. In modern algebra categories can be defined between such algebraic structures as groups, rings and modules. Therefore objectives of this paper are:

1) To give the history which lead to the development category concepts.

2) To cite, describe and give examples of some basic and/or universal properties of categories.

3) To define and give examples of categories and functors for algebraic structures such as semi-groups, groups, ring and lattices.

4) To cite and give proofs of some special properties of algebraic categories and functors.

5) To illustrate how some mathematical structures are equivalent using categories.

6) To cite and prove theorems in one mathematical area that were proved due to the resources of category theory.

Definition of Categories 1.1

A category is determined by having $\mathcal{C}$, a class of objects $A, B, C \ldots$ where there is a set of mappings or morphisms between any disjoint pair of objects of $\mathcal{C}$ such that there are functions $F: A \rightarrow B$ in the set of hom $(A, B)$. Also, there must be for any triple $(A, B, C)$ of $\mathcal{C}$ a composition map of morphisms: $\hom(B, C) \times \hom(A, B) \rightarrow \hom(A, C)$. (or with functions $f: A \rightarrow B$, $g: B \rightarrow C$ and $h: A \rightarrow C$
such that $g \circ f = h$). This composition of morphisms is subject to two axioms:

1) **Associativity of composition**

$$(h \circ g) \circ f = h \circ (g \circ f)$$

2) **Identity of composition**

For any object $B$, there is the identity $1_B: B \to B$ such that for $f: A \to B$ and $g: B \to C$, $1_B \circ f = f$ and $g \circ 1_B = g$.

The above definition uses the term "class" that refers to "large collections". It must be noted that every set is a class but not every class is a set. But how is it every class is not a set? To handle the idea of class more clearly let us cite the following mathematical characteristics of a class.

Let $U = \{ x/ x \text{ is a set and } p(x) \}$ be a collection whose members $X$ are exactly those sets with property $P$. This "large collection" is called the class of all sets with property $P$. Therefore "Classes" are considered to be the "subcollections" of the class $U$ (the universe) of all sets. By regarding each member of a set as a set, sets can be considered special classes. Those classes which are not sets are called proper classes. These are classes that can not be classified as a set because they are not subsets of themselves.

Given classes $A$ and $B$, we can form the following classes:
a) The union class \( A \cup B = \{ x \in \text{A} \text{ or } x \in \text{B} \} \).

b) The intersection class \( A \cap B = \{ x \in \text{A} \text{ and } x \in \text{B} \} \).

c) The complement class \( A - B = \{ x \in \text{A} \text{ and } x \notin \text{B} \} \).

d) The cartesian product class \( \text{A} \times \text{B} = \{ (a,b) | a \in \text{A} \text{ and } b \in \text{B} \} \).
e) The disjoint union class \( \text{A} \uplus \text{B} = \{ \text{A} \times \{ a \} \cup \text{B} \times \{ b \} \} \).

Examples of Classes:

a) Class of all groups
b) Class of all counting numbers
c) Class of all rings
d) Class of all vector spaces

Note that the class of all vector spaces is not a set, since this class is not a vector space.

Examples of Categories

1) Let \( \mathcal{C} \) be the class of all topological spaces. If the objects are topological spaces and the morphisms are continuous maps, the topological spaces form a category.

2) Let \( \mathcal{S} \) be the class of all sets. Let morphisms from set \( A \) to set \( B \) be subsets of \( A \times B \). Define the composition of relations as follows:

\( f \in A \times B, \ g \in B \times C \), then \( gf = \{(a,c)\} \) if there is a \( b \in B \) such that \((a,b) \in f, (b,c) \in g\). Thus sets and relations form a category.
3) Let a class of objects be complex Banach spaces and morphisms be bounded linear transformations. Thus complex Banach spaces and their linear transformations form a category.
Chapter II

Fundamental Properties of Categories and Functors
A. Introduction

Category theory has a unique historical pattern of development for it was formulated in order to solve problems in algebraic topology and has grown since then to be a mathematical field in its own right while sparking the formulation of "universal" algebra, being used to characterize notions and concepts of topology, analysis and geometry and "vitally altered" logic and the foundations of mathematics. These are amazing accomplishments in the light of only a historical period of forty years!

The purpose of this chapter is to, intuitively and technically, take a glimpse at three powerful properties that extend throughout category theory and make category theory a "unifying force of mathematics. The following properties apply to all of categories and are presented here to enhance understanding of algebraic categories:

B. Categorical Maps

Morphisms

The essence of category theory is in its morphisms which allows for the comparison of mathematical structures. Therefore, "... in general category theory the main consideration is the morphisms and how they are composed; the objects serve little purpose other than to remind us of the domain and range of the morphisms; and elements of objects are not mentioned at all."¹ "Because of the one - to

¹Horst Herrlick and George E. Strecker, Category Theory (Boston, Mass.: Allyn and Bacon Inc., 1973) p. 4.
one correspondence $A \leftrightarrow \mathfrak{C}$ between $\mathfrak{C}$ - objects and $\mathfrak{C}$ - identity morphisms in any category $\mathfrak{C}$, we will be able ... to provide an "object - free" definition of category which is equivalent to our earlier definition".  

Definition (a second definition of category) 2.1

A category is a pair $(M, \circ)$ where $M$ is a class and $\circ$ is a partial operation on $M$ satisfying the following conditions:

1. Matching Condition: For all $f, g, h \in M$, if $f \circ g$ and $g \circ h$ are defined then $f \circ (g \circ h)$ is defined and $(f \circ g) \circ h$ is defined.

2. Associativity Condition: For all $f, g, h \in M$, $f \circ (g \circ h)$ is defined, and when they are defined they are equal.

3. Identity Existence Condition: For every $f \in M$, there exist morphisms $l_c$ and $l_d$ which are identities with respect to $\circ$ such that $l_c \circ f$ and $f \circ l_d$ are defined.

4. Smallness of Morphism Class Condition: For all identities $l_c$ and $l_d$ in $M$, the class of elements $f \in M$ such that $f \circ l_c$ and $l_d \circ f$ are defined is a set.

There are several kinds of morphisms (a mapping between sets with underlying functions) that are analogous to several special types of functions. Those correspondence are:

---

Functions          Morphisms
injective functions  \rightleftharpoons \text{monomorphisms}
surjective functions  \rightleftharpoons \text{epimorphisms}
bijective functions  \rightleftharpoons \text{isomorphisms}
identity functions   \rightleftharpoons \text{automorphisms}

Since the terminology of morphism appears frequently in the theory of categories and functors some morphisms properties and definitions involving categories are below:

Proposition 2.2

If \( f: A \to B \) is a morphism in a category \( \mathcal{C} \) then the following are equivalent:

(1) \( f \) is a monomorphism in \( \mathcal{C} \)

(2) \( f \) is said to be a section in \( \mathcal{C} \) provided that \( \exists \) some \( \mathcal{C} \)-morphism \( f: B \to A \) such that \( f \circ 1_B = 1_A \).

(3) \( f \) is left cancellable with respect to composition in \( \mathcal{C} \) if for all \( \mathcal{C} \)-morphisms \( h \) and \( k \) such that \( f \circ h = f \circ k \) it follows that \( h = k \).

Proposition 2.3

If \( f: A \to B \) is a morphism in a category \( \mathcal{C} \) then the following are equivalent:

(1) \( f \) is an epimorphism in \( \mathcal{C} \)

(2) \( f \) is said to be a retraction in \( \mathcal{C} \) provided that \( \exists \) some \( \mathcal{C} \)-morphism \( g: B \to A \) such that \( f \circ g = 1_B \).

(3) \( f \) is right cancellable with respect to composition in \( \mathcal{C} \) if for all \( \mathcal{C} \)-morphisms \( h \) and \( k \) such that \( h \circ f = k \circ f \), it follows
that \( h = k \).

**Proposition 2.4**

If \( f: A \rightarrow B \) is a morphism in a category \( \mathcal{C} \), then the following are equivalent:

1. \( f \) is an isomorphism in \( \mathcal{C} \)
2. \( f \) is both a section and a retraction in \( \mathcal{C} \)
3. For \( g: B \rightarrow A \) in \( \mathcal{C} \), then \( g \circ f = 1_A \) and \( f \circ g = 1_B \).

**Proposition 2.5**

If \( f: A \rightarrow B \) and \( g: B \rightarrow C \) are sections in category \( \mathcal{C} \), then \( g \circ f : A \rightarrow C \) is a section.

**Proposition 2.6**

If \( f: A \rightarrow B \) and \( g: B \rightarrow C \) are \( \mathcal{C} \) retractions, then \( g \circ f : A \rightarrow C \) is a \( \mathcal{C} \) retraction.

**Proposition 2.7**

In any category, the composition of isomorphisms is an isomorphism.

**Definition 2.8**

An object \( A \) of a category \( \mathcal{C} \) is said to be isomorphic with an object \( B \) of \( \mathcal{C} \) provided that there exist some \( \mathcal{C} \) - isomorphism \( f: A \rightarrow B \).

**Proposition 2.9**

For any category \( \mathcal{C} \), "is isomorphic with" yields an equivalence relation on objects of \( \mathcal{C} \).
Functors

The foundation for comparing categories includes the concept of the functors ——-a "structure preserving function" between categories.

Definition 2.10

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(\mathcal{C}, F, \mathcal{D})$ where $F$ is a function from the class of morphisms of $\mathcal{C}$ to the class of morphisms of $\mathcal{D}$. $(F: \text{mor } \mathcal{C} \rightarrow \text{Mor } \mathcal{D})$ satisfying the following conditions:

1. $F$ preserves identities: if $1_A$ is a $\mathcal{C}$ identity, then $F(1_A)$ = or $F(1_A)$ is a $\mathcal{D}$ identity.

2. $F$ preserves composition: $F(f \circ g) = F(f) \circ F(g)$ or whenever $\text{dom } (f) = \text{codom } (g)$, then $\text{dom } F(f) = \text{Cod } (F(g))$ and the above equality holds.

Note: (a) for all $\mathcal{C}$ - objects $A$ and $B$

$F(\text{hom } \mathcal{C}(A, B)) = \text{hom } (F(A), F(B))$.

(b) For any functor $G: \mathcal{C} \rightarrow \mathcal{D}$ can easily be recovered from its "object - function"

$F: \mathcal{C}$ - objects $\rightarrow \mathcal{D}$ - objects and all of the restrictions:

$F(\text{hom } (FA, FB)) = \text{hom } (A, B)$.

Often functors are described by thier "hom-set" restrictions.

Examples of Functors 2.11

(1) For any category $\mathcal{C}$, $(\mathcal{C}, 1 \text{ mor } \mathcal{C}, \mathcal{C})$ is a functor on $\mathcal{C}$ and denoted by $1_{\mathcal{C}}$. 
(2) If \( \mathcal{C} \) is a subcategory\(^3\) of \( \mathcal{D} \) and \( E: \text{Mor}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{D}) \) is the inclusion function, then \( E: \mathcal{C} \rightarrow \mathcal{D} \) is a functor, called the inclusion functor from \( \mathcal{C} \) to \( \mathcal{D} \).

(3) Let the object of \( \mathcal{C} \) be sets with a certain structure (group, topological spaces etc.)

Let the morphisms be structure-preserving maps (homomorphisms, continuous maps etc). Then the forgetful functor \( U: \mathcal{C} \rightarrow \text{Set} \) assigns to each object its underlying set and to each morphism the corresponding set map.

Other forgetful functors forget only part of their categorical structure. Example \( U: \text{Ring} \rightarrow \text{Ab} \) (\( \text{Ab} \) is the category of abelian groups of \( R \) and to each morphism \( f: \text{Rings} \rightarrow \text{Group} \) where the multiplication is forgotten.

(4) The function \( F: \text{Set} \rightarrow \text{Group} \) that assigns to any set \( A \) the free group with set of generators \( A \), and to each set function \( f \) the induced homomorphism that coincides with \( f \) on the generators, is a functor, called the free group functor.

(5) Let \( \mathcal{C} \) be the category of let \( A \)-modules and \( X \) a fixed right \( A \)-module. We obtain a functor \( T_X \) from the category \( \mathcal{C} \) into the \( \text{Ab} \) category (abelian groups) by setting \( T_X(Y) = Y \otimes_A X \) for any homomorphism \( U: Y \rightarrow Y' \) of the \( A \)-module \( Y \) into the \( A \)-module

\(^3\)Subcategories are defined in the \( C \) section of this chapter.
\( Y, \mathcal{T}_x (u) \) is the map from \( Y \otimes_A X \) into \( Y' \otimes_A X \) induced by \( U \).

The following diagram illustrates the above functor:

\[
\begin{array}{ccc}
\mathcal{T}_x (Y) & \to & Y \otimes_A X \\
\downarrow & & \downarrow \\
\mathcal{T}_x (Y') & \to & Y' \otimes_A X
\end{array}
\]

Natural Transformations

"Natural transformations can be regarded as morphisms between functors."\(^4\)

Definition

Let \( S: \mathcal{C} \to \mathcal{D} \) be a functor. Let \( T: \mathcal{C} \to \mathcal{D} \) be a functor

A natural transformation \( \eta: S \to T \) is a map which assigns to each object \( A \) of \( \mathcal{C} \) objects a morphism \( \eta_A: S(A) \to T(A) \) in \( \mathcal{D} \) in such a way that for every morphism \( f: A \to B \)

The following diagram is commutative

\[
\begin{array}{ccc}
S(A) & \xrightarrow{\eta_A} & T(A) \\
\downarrow & & \downarrow \\
S(B) & \xrightarrow{\eta_B} & T(B)
\end{array}
\]

\(^4\text{Herrlich and Strecker, } \textit{Category Theory}, \text{ p. 77}\)
Also \( T(f) \circ \iota_A = \iota_B \circ S(f) \) for all \( f : A \to B \) in \( \mathcal{C} \).

Examples of Natural Transformations 2.12

1. For each group \( A \), let \( A' \) be the commutator subgroup of \( A \); i.e. the subgroup generated by all elements of \( A \) of the form \( ghg^{-1} h^{-1} \).

Defined \( F : \text{Group} \to \text{Group} \) by \( F(A) = A' \) and \( F : F(A) \to B \) is the commutator subgroup functor. There is a natural transformation from the commutator subgroup functor \( F \):

\[
F : \text{Group} \to \text{Group} \to \text{Group} \to \text{Group}
\]

2. Let \( U : \text{Group} \to \text{Set} \) be the underlying functor. Let \( F \) be the free functor from \( \text{Set} \) to \( \text{Group} \). There exist natural transformations

\[
\eta = (\eta_A) : \text{Set} \to \text{U of Group}
\]

\[
\varepsilon = (\varepsilon_B) : F \circ U \to \text{I of Group}
\]

where \( \eta_A : A \to U(F(A)) \) is the insertion of the generator of \( F(U(B)) \) and \( \varepsilon_B : F(U(B)) \to B \) is the unique group homomorphism (preserves multiplication) induced by the identity function on \( U(B) \).

C. Unique properties of categories and functors

Categories and functors have wide reaching consequences in mathematics because of their duality, equivalence and universality properties. With duality of functors and categories we have two categories or two functors for the price of one. Under the principles of equivalence "difficult problems in some areas of mathematics can

\[5\text{Ibid, p. 5.}\]
Duality of Categories and Functors

For every category \( \mathcal{C} = (0, M, \text{dom}, \text{cod}, \circ) \) there exists \( \mathcal{C}^{\text{op}} \) (the opposite or dual category) = \( (0, M, \text{cod}, \text{dom}, \circ^\text{op}) \) where \( \circ^\text{op} \) is defined by \( f \circ^\text{op} g = g \circ f \). \( \mathcal{C} \) and \( \mathcal{C}^{\text{op}} \) have the same objects and morphisms but their domain and codomain have been interchanged or reversed.

Every categorical statement or concept has a dual. If \( \mathcal{C} \) has a property \( P \), then \( P^{\text{op}} \) can be determined by taking the same objects, morphisms and compositions and reversing the arrows.

Proposition 2.13

The opposite category of a category is a category.

Proposition 2.14

If \( \mathcal{C} \) is any category, then \( (\mathcal{C}^{\text{op}})^{\text{op}} = \mathcal{C} \).

Proposition 2.15

If \( S \) is a categorical statement which holds for all categories then \( S^{\text{op}} \) also holds for all categories.

Examples of Dual categories 2.16

(1) If \( \mathcal{C} \) is the category associated with an ordered set, then \( \mathcal{C}^{\text{op}} \), is the oppositely ordered set (\( \leq \) is replaced by \( \geq \)).

(2) If \( \mathcal{C} \) is a monoid or a group or a ring, then \( \mathcal{C}^{\text{op}} \) is the opposite monoid, group or ring.

Definition 2.16

If \( p \) is a categorical property (concept) then if \( P = p^{\text{op}} \) is called self-dual.
If a $F: \mathcal{C} \to \mathcal{D}$ is involved in a categorical statement or property, then in the dualization process the functor becomes $F^{\text{op}}: \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$. Under dualization categories and functors the dual concepts carry the prefix co to indicate its duality. Examples (without giving definitions) are products and co-products, limits and co-limits and well-powered and co-well powered. Many, many properties of categories and functor can be proved using duality.

**Equalivalence of Categories**

In group theory two or more groups are structurally the same if they are isomorphic to each other. This is a strong concept. In category theory to determine "structural sameness" of categories we use the weaker concept of "equivalence" instead of isomorphism.

"The category $F - \text{mod}$ of all vector spaces over a field $F$ and its full subcategory consisting of all objects of the form $F^I$ (that is all powers of the field $F$) are obviously not the same nor even isomorphic. Yet, from a categorical point of view, ..., they have the same categorical characteristics. The main difference between the two lies in the fact that in any two isomorphic objects are identical, whereas there exists many different objects isomorphic to any given object in $F - \text{mod}$. Below we will define "equivalence of categories" in such a way that two categories $\mathcal{A}$ and $\mathcal{B}$ will be equivalent provided that the only difference between them lies in the fact that in one of them some object might be

---

6 Full subcategory is defined in section 9 of this chapter.
"counted" more times than in the other — in other words, provided that the categories obtained from A and B by counting each object just once, are isomorphic."\(^7\)

Definition 2.17

(a) A functor \(F : A \rightarrow B\) is said to be an isomorphism from A to B provided that there exists a functor \(G : B \rightarrow A\) such that \(G \circ F = 1_A\) and \(F \circ G = 1_B\).

(b) Categories \(A\) and \(B\) are said to be isomorphic (denoted \(\cong\)) provided there is an isomorphism between them.

Examples of Isomorphic Categories 2.18

(1) Abelian group Category and \(\mathbb{Z}\) - Module category are isomorphic.
(2) Every ring \(R\), \(R\) - module category is isomorphic with Mod-\(R^*\) where \(R^*\) is the dual of \(R\).

Definition 2.19

(a) A category \(\mathcal{C}\) is called skeletal provided that isomorphic \(\mathcal{C}\) - objects are identical.

(b) A skeleton of a category \(\mathcal{C}\) is a maximal full skeletal subcategory of \(\mathcal{C}\).

\(^7\) Herrlich and Strecker, Category Theory, p. 87.
Proposition 2.20

Every category $\mathcal{C}$ has a skeleton.

Proposition 2.21

Any two skeletons of a category are isomorphic.

Definition 2.22

A category $\mathcal{A}$ is said to be equivalent to a category $\mathcal{B}$ (denoted by $\mathcal{A} \sim \mathcal{B}$) if and only if $\mathcal{A}$ and $\mathcal{B}$ have isomorphic skeletons.

Proposition 2.23

Skeletal categories are equivalent if and only if they are isomorphic.

Definition 2.24

A functor $F: \mathcal{A} \to \mathcal{B}$ is called an equivalence if there is another functor $G: \mathcal{B} \to \mathcal{A}$ such that $F \circ G \simeq 1_{\mathcal{B}}$ and $G \circ F \simeq 1_{\mathcal{A}}$. The equivalence of $F: \mathcal{A} \to \mathcal{B}$ implies that categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent.

Examples of Equivalent Categories 2.25

(1) For any field $F$, the category of finite dimensional vector spaces over $F$ is equivalent to the category of $F$-matrices.

(2) Let $R$ be the category of reflexive Banach spaces and norm-decreasing linear transformations. The functor $\text{Hom}(\cdot, R)$: $R^{\text{op}} \to R$ that sends each reflective Banach space to its conjugate space is an equivalence, but not an isomorphism.
(5) Every isomorphism is an equivalence.

Universality

The universal map and adjoints are used to generate generalizations about different objects and different morphisms in different areas of mathematics.

Definition 2.26

Let \( G: A \to B \) be a functor and let \( B \in A \) - objects. A pair \((U, A)\) with \( A \in A \) - objects and \( V: B \to G(A) \) is called a universal map for \( B \) with respect to \( G \) provided that for each \( A' \in A \) - objects and each \( f: B \to G(A) \) there exists a unique \( A \) - morphism \( f: A \to A' \) such that the triangle

\[
\begin{array}{c}
B \\
\downarrow f \\
G(A) \\
\downarrow G(f) \\
G(A') \\
\downarrow f \\
A'
\end{array}
\]

Commutes.

Dually: If \( F: A \to B \) is a functor and \( B \in B \) - objects, then a pair \((A, U)\) is called a co-universal map for \( B \) with respect to \( F^{op}: A^{op} \to B^{op} \) that is provided that \( U: F(A) \to B \) and for each \( A \) - object \( A' \) and each \( B \) - morphism morphism \( f: A' \to A \) such that the triangle:

\[
\begin{array}{c}
A' \\
\downarrow f \\
F(A') \\
\downarrow f \\
F(A) \\
\downarrow f \\
B \\
\end{array}
\]

Commutes.
Examples of universal maps 2.27

(1) Let \((A, U)\) be the concrete category (has underlying sets) of Groups. Then for each set \(B\) there exists a \(U\) - universal map \((u_B, A_B)\). \(A_B\) is the free group generated by \(B\). \(U_B: B \rightarrow U(A_B)\).

(2) Let category \(\mathbf{A}\) be the category of \(R\) - Modules (\(R\) is a commutative ring). Let \(\text{Hom}(G, -): \mathbf{A} \rightarrow \mathbf{A}\) be the internal \(\text{Hom}\) - functor with respect to the \(\mathbf{A}\) - Object \(C\) and let \(B \in \mathbf{A}\) - objects. So \(A_B = B \otimes_R C\) and \((u_B(b)) (c) = b \otimes c\) or \(U_B: B \rightarrow \text{Hom}(C, A_B)\) is a universal map.

Adjoint functors

Another notion that is used to generate generalizations that cover many areas is the adjoint. This notion resembles an inverse.

Definition 2.28

(1) If \(A\) and \(B\) are categories, \(G\) and \(F\) are functors and \(\eta\) and \(\epsilon\) are natural transformations such that:

(i) \(G: A \rightarrow B\) and \(F: B \rightarrow A\)

(ii) \(U: I_B \rightarrow G \circ F\) and \(\epsilon: F \circ G \rightarrow I_A\)

(iii) \(G_1: \xrightarrow{\eta \star G} G \circ F \circ G_1 \xrightarrow{G_1 \star \epsilon} G_1 = G \xrightarrow{I_G} G_1\)

\(F \xrightarrow{F \star \eta} F \circ G \circ F \xrightarrow{\epsilon \star F} = F \xrightarrow{I_F} F\)

then this is called an adjunction or adjoint situation, and is denoted by \((\eta, \epsilon): F \rightarrow G\) or \(\text{mor}\). briefly by \((\eta, \epsilon): F \rightarrow G\).
(2) If \((\eta, \xi)_E \): \(F \rightarrow G\), then \(F\) is said to be a left adjoint of \(G\), \(G\) is said to be a right adjoint of \(F\), \(\eta\) is called the unit of the adjunction and \(\xi\) is called the counit of the adjunction.

(3) A functor \(G: A \rightarrow B\) is said to have a left adjoint provided that \(\exists \eta \) and \(\xi \in \{G, \eta \}: F \rightarrow G\). In other words, a functor has a left adjoint provided it is the right adjoint of some functor, and it has a right adjoint provided that is the left adjoint of some functor.

The following theorems show the relationship between the universal map and the adjoint functor.

Proposition 2.29

Let \(G: A \rightarrow B\) be a functor such that for each \(B \in B\)-objects there exists a \(G\)-universal map \((\eta_B, \xi_B)\).

(1) Then there exists a unique functor \(F: B \rightarrow A\) such that

(a) for each \(B \in B\)-objects, \(F(B) = A_B\); and

(b) \(\eta = (\eta_B): 1_B \rightarrow G \circ F\) is a natural transformation.

(2) There is a unique natural transformation

\[
\begin{align*}
\xi : F \circ G & \rightarrow 1_A \\
F \circ G & \rightarrow G \\
G \circ F & \rightarrow F \\
G & = G
\end{align*}
\]

that is for each \(A \in A\)-objects,

\[G(F_A) \circ \eta_A = 1_{G(A)} \text{, and} \]

For each \(B \in B\) - objects

\[\xi F(B) \circ F(\eta_B) = 1_{F(B)} \text{.} \]

Proposition 2.30

Let \(G: A \rightarrow B\).
(1) If each $B \in B$-objects has a $G$-universal map $(\eta_B, \lambda_B)$, then there exists a unique adjoint situation $(\eta, \epsilon) : F \rightarrow G$ such that $\eta = (\eta_B)$ and for each $B \in B$-objects, $F(B) = A_B$.

(2) Conversely, if we have an adjoint situation $(\eta, \epsilon) : F \rightarrow G$, then for each $B \in B$-objects, $(\eta_B, \epsilon(B))$ is a $G$-universal map for $B$.

Examples of adjoint Functors 2.31

(1) Let $\mathcal{A}$ = Group category and $\mathcal{B} = Set$ category. If their left and right adjoints are a free group functor and forgetful functor respectively.

(2) Let $\mathcal{A} = Ring$ category and $\mathcal{B} = abelian group$ category. If $G: \mathcal{A} \rightarrow \mathcal{B}$ then left and right adjoints are a tensor ring functor and a forgetful functor respectively.

Subcategories

The work subcategory is analogous to the words subset, subgroups, submodule etc. Therefore one would expect the following definition of subcategory.

Definition 2.32

A category $\mathcal{B}$ is said to be a subcategory of the category $\mathcal{C}$ provided that the following conditions are satisfied:
(1) B-objects $\subseteq$ C-objects

(2) B morphisms $\subseteq$ C-morphisms

(3) The domain, co-domain and composition functions of B are restrictions of the corresponding functions of C.

(4) Every B identity in a C-identity

(5) $\text{hom}_B(A,B) \subseteq \text{hom}_C(A,B)$.

Proposition 2.33
A subcategory is a category.

Definition 2.34
A subcategory $B$ of category $C$ is said to be a full subcategory of $C$ provided that for all $A, B$ objects of $B$, $\text{hom}_B(A, B) = \text{hom}_C(A, B)$.

Examples of full subcategories 2.35
(1) Each category is a full subcategory of itself.

(2) The category of finite sets is a full subcategory of set (category).

(3) The commutative groups determine a full subcategory of the category of all groups. Also there is the full subcategory of free groups; and there is the full subcategory of free abelian groups.
Chapter III

Algebraic Categories

and Functors
Introduction

This chapter has two purposes:

1) to give some definitions and explanations as to how algebraic categories differ from other types of categories.

2) to give examples of algebraic categories that make the difference between algebraic categories and other categories as concrete as possible.

Algebraic Categories and Functors

Since the concepts of categories and functors appeared in the early forties attempts have been made to give a general description of algebraic categories such as rings, lattices, groups and $R$-modules etc. At first the attempt to study those categories came under the umbrella of "universal algebra" with definitions involving "algebraic theories". Those definitions are cumbersome and not general enough to include categories such as compact Hausdorff spaces and complete boolean algebras. Those definitions may be quoted as follows:

"Definition 1 an algebraic theory is a category $A$ with the following properties:

(i) $|A|$ consists of countably many different objects $A^0, A^1, A^2, A^3, \ldots$.

(ii) For $k \geq 0$, $A^k$ is a product of $A^1$ with itself $K$-times with projections $p_1^k, p_2^k, \ldots, p_k^k$ for $K \geq 1$. Here
$p^1_1 = 1_A$ is assumed. The morphisms $A^n \rightarrow A^1$ are called $n$-ary operations. A theory-morphism $F : A \rightarrow B$ is a functor satisfying (1) $F(A^n) = B^0$ and $F(p^k j) = p^k j$ for all $(k, j)$ with $1 \leq j \leq k < \infty$. The algebraic category corresponding to $A$ is the full subcategory $\mathcal{A}$ of $(A, \text{sets})$ whose objects are the functors which preserve finite products. The objects of $\mathcal{A}^b$ are called $\mathcal{A}$-algebras, the morphisms are called $\mathcal{A}$-homomorphisms.\footnote{Schubert, Horst, \textit{Categories}; New York; Springer-Verlag, 1972, p. 222.}

In the above definition the projection morphisms refer to maps of $\prod_{i \in I} A_i \rightarrow A_i | i \in I$ where cartesian product $\prod_{i \in I} A_i$ equals $A_1 \times A_2 \times \ldots \times A_n$ when $I = \{1, \ldots, n\}$. Notice since $1 \leq j \leq k < \infty$, this definition has functors which preserve finite products. Further generalizations of the stages of the projections allows for the creation of further "algebraic theories 1)" and "algebraic categories". If the stages become uncountable $\sigma$-complete boolean algebras, and $\kappa$ complete lattices become algebraic categories. Therefore one may conclude the "universal algebra" definition of algebraic categories is insufficient.
Dropping the ideas involving preserving finite products, a more general axiomatic approach can be taken to define algebraic categories.

Before algebraic categories can be defined some preliminary definitions and propositions are needed.

Definition 3.1

A category is said to be concrete if it is isomorphic with a subcategory of the category of sets. That is \( C = (0, U, \text{hom}) \) and

(a) \( 0 \) is a class of objects
(b) \( U : 0 \rightarrow \text{Set} \) is a set valued function where for each \( a \in U \), \( U(A) \) is called the underlying set of \( A \).
(c) \( \text{hom}: 0 \times 0 \rightarrow \text{Set} \) is a set valued function where for each pair \( (A, B) \) of \( C \) - category objects, \( \text{hom}(A, B) \) is called the set of all \( C \) - category morphisms

Examples of concrete categories 3.2

(1) Group and their homomorphisms
(2) Rings and their homomorphisms
(3) Right \(-\) \( R \) modules and their homomorphisms
(4) Semi \(-\) groups and their homomorphisms
(5) Lattices and their homomorphisms

Definition 3.3

A function \( F : C \rightarrow D \) is said to preserve a categorical property \( P \) provided that the image under \( F \) of each morphism in \( C \) with property \( P \) has property \( P \) in \( D \).
(1) $c: B \rightarrow C$ is a $C$-morphism.

(2) $c \circ f = e \circ g$.

(3) For $C$-morphism $c': B \rightarrow C$ such that $c' \circ f = c' \circ g$, there exist a unique $C$-morphism $\overline{c}: C \rightarrow C'$ such that the triangle commutes.

Definition 3.8

(1) If $(h_i)_{i \in I}$ is a non-empty indexed family of morphisms contained in $\text{hom}_C(A, B)$, then $(E, e)$ is said to be a multiple equalizer of $(h_i)_{i \in I}$, denoted by $(E, e) \approx \text{Equ} \left( (h_i)_{i \in I} \right)$, provided that:

(i) $e: E \rightarrow A$

(ii) For all $i, j \in I$, $h_i \circ e = h_j \circ e$.

(iii) If $c': E' \rightarrow A$ such that $h_i \circ c' = h_j \circ c' \forall j \in I$, there is a unique morphism $\overline{c} \rightarrow c' \circ \overline{\iota} = c'$.

(2) A category $C$ has multiple equalizers provided that each non-empty indexed family of morphisms that have a common domain and a common codomain has a multiple equalizer.

Definition 3.9

A subobject of an object $b \in \text{Ob}(C)$ is a pair $(A, f)$ where $f: A \rightarrow B$ is a monomorphism.
Definition 3.10

If \( \mathbf{c} : \textbf{B} \rightarrow \textbf{C} \) is a \( \mathbf{C} \) - morphism then \((\mathbf{c}, \mathbf{C})\) is called a regular quotient object of \( \textbf{B} \) and \( \mathbf{C} \) is called a regular epimorphism iff \( \exists \) \( \mathbf{C} \)-morphisms \( f \) and \( g \Rightarrow (\mathbf{c}, \mathbf{C}) \Rightarrow \text{Equ} \) \( (f,g) \).

Definition 3.11

A concrete category \((\mathbf{C}, \mathbf{U})\) is called algebraic provided that it satisfies the following three conditions:

1. \( \mathbf{A} \) has coequalizers
2. \( \mathbf{U} \) has a left adjoint
3. \( \mathbf{U} \) preserves and reflects regular epimorphisms.

Examples 3.12

There are many properties and theorems not cited here that make it easy see that the following categories with their forgetful functors are algebraic categories. The categories of semigroups, groups, rings and \( \mathbb{R} \)-modules along with their forgetful functors are examples of algebraic categories.

It has been established in a previous chapter that the forgetful functor is a universal map that produces a corresponding adjoint functors. Listed below are categories and their left adjoint functors:
<table>
<thead>
<tr>
<th>Categories</th>
<th>Left Adjoint</th>
<th>forgetful functors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Semi-Group</td>
<td>U: S Grp $\rightarrow$ Set</td>
<td>X $\rightarrow$ FX</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Free Semi Group generated by X)</td>
</tr>
<tr>
<td>2. Group</td>
<td>U: Grp $\rightarrow$ Set</td>
<td>X $\rightarrow$ F X</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Free group, generated by X)</td>
</tr>
<tr>
<td>3. Ring</td>
<td>U: Ring $\rightarrow$ Mon</td>
<td>M $\rightarrow$ $\mathbb{Z}(M)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(integral monoid ring)</td>
</tr>
<tr>
<td>4. R-module</td>
<td>U: R-Mod $\rightarrow$ Set</td>
<td>X $\rightarrow$ F X</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Free R-Module, basis X)</td>
</tr>
<tr>
<td>5. R-Module</td>
<td>U: R-Mod $\rightarrow$ Abel</td>
<td>A $\rightarrow$ R$\times$ A</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(tensor Product)</td>
</tr>
</tbody>
</table>

Forgetful functors have properties that can be used to help verify that a category and its forgetful functor is an algebraic category. Those properties are:

**Proposition 3.13**

Concrete categories by definition have forgetful functors.

**Proposition 3.14**

Forgetful functors are faithful since each hom set restriction of F is injective (one-to-one).

**Proposition 3.14**

Every faithful functor $F: \mathcal{A} \rightarrow \mathcal{B}$ reflects monomorphisms,
epimorphisms, bimorphisms etc. (proof p. 70 categories Herrlich and Strecker)

Proposition 3.16

Every functor preserves identities, isomorphisms, sections, retractions and commutative triangles.

Proposition 3.17

A forgetful functor reflects regular epimorphisms

In all of the above example categories can be established to have co-equalizers. The following proof establishes co-equalizers for the category of Groups.¹

Proof:

Let \( f, g : X \rightarrow Y \) be group homomorphisms and define \( R_0 \subseteq Y \times Y \) by \( R_0 = \{ (f(x), g(x)) | x \in X \} \). The set \( \overline{R} = \{ \overline{r} | r \in R \} \) is a group congruence on \( Y \). The set \( R_0 \subseteq \overline{R} \). Now let \( h : Y \rightarrow Z \) be a group homomorphism such that \( h \circ f = \lambda \circ g \).

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{\lambda} \\
Y & \xrightarrow{\pi} & Y/\overline{R} \\
\end{array}
\]

¹In the appendix of this paper are proofs that establish \((G,U)\) as an algebraic category.
As $R$ is a group congruence $R \subseteq R$. Therefore $\psi \{ y \} = h(y)$ well defined, is easily checked to be a group homomorphism and is unique. Thus $\mathcal{N}_R: Y \to Y/R = \text{coeq } (f, g)$. 

$$R_0 \subseteq R = \{ (y_1, y_2) \mid h(y_1) = h(y_2) \}$$
Chapter IV

Some Special Properties of Algebraic Categories and Functors
Introduction

There are some specific properties of categories that are also specific properties of algebraic categories. Here, definitions will be given that describe those specific properties:

Definition 4.1:

A product of a pair \((A, B)\) of \(\mathcal{C}\)-objects is a triple \((P, \pi_A, \pi_B)\) where \(P\) is a \(\mathcal{C}\)-object and \(\pi_A : P \to A\), \(\pi_B : P \to B\) are \(\mathcal{C}\)-morphisms (called projections with the property that if \(C\) is any \(\mathcal{C}\)-object and \(f : C \to A\), \(g : C \to B\) are arbitrary \(\mathcal{C}\)-morphisms, then \(\exists\) a unique \(\mathcal{C}\)-morphism usually denoted by \(\langle f, g \rangle : C \to P\) such that the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\langle f, g \rangle} & P \\
\downarrow & & \downarrow \\
A & \xrightarrow{\pi_A} & B
\end{array}
\]

Commutes.

Examples of Product Mappings 4.2

<table>
<thead>
<tr>
<th>Category</th>
<th>(A \times B)</th>
<th>((P, \pi_A, \pi_B))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sets</td>
<td>Cartesian product</td>
<td>disjoint union</td>
</tr>
<tr>
<td>Group</td>
<td>direct product</td>
<td>free objects or universal objects</td>
</tr>
</tbody>
</table>

Definition 4.3:

A source in \(\mathcal{C}\) is a pair \((X, (f_i)_{i \in I})\), where \(X\) is a \(\mathcal{C}\)-object and \(f_i : X \to (X_i)_i\) is a family of \(\mathcal{C}\)-morphisms each with domain
X. In this case X is called the domain of the source and the family \((X_i)_I\) is called the codomain of the source.

**Definition 4.4:**

A source \((X, F_i)\) is called a mono source provided that the \(F_i\) can be simultaneously cancelled from the left. That is for any pair \(\begin{array}{c} Y \xrightarrow{f} X \\end{array}\) of morphisms such that \(F_i \circ r = f_i \circ s\) for each \(i\), it follows that \(r = s\).

**Examples of 4.5:**

Let \((F_i, X)\) be a sink in the category of groups. Then \((F_i, X)\) is a sink and iff the union of the set images of the homomorphisms \(f_i\) generates \(X\).

The limit functor is a generalization of the concepts of "equalizers," "products," etc.

**Definition 4.6:**

If \(I\) and \(\mathcal{C}\) are categories and \(D: I \rightarrow \mathcal{C}\) is a functor, then a natural source of \(D\) is a source \((L, (i_i)_{i \in \text{Ob}(I)})\) in \(\mathcal{C}\) such that for each \(i \in \text{Ob}(I)\), \(\ell_i: L \rightarrow D(i)\); and for all morphisms \(m: i \rightarrow j\) in \(I\), the triangle

\[
\begin{array}{ccc}
L & \xrightarrow{\ell_i} & D(i) \\
\downarrow^{\ell_j} & & \downarrow^{D(m)} \\
D(j) & \xrightarrow{D(m)} & D(j)
\end{array}
\]

Commutes.

"In other words, if \(L: I \rightarrow \mathcal{C}\) is the constant functor whose value at each object is \(L\) and whose value at each morphism is \(L_m\),
and if \((L_i \mid i \in I)\) is a source in \(C\), then \((L_i \mid i \in I)\)
is a natural source for \(D\) if and only if \((L_i \mid i \in I)\) is a natural
transformation from \(L\) to \(D\).\(^1\)

Definition 4.7:

A category is \(I\)-complete when the category \(I\) exists and
every functor \(D : I \to C\) has a limit.

Definition 4.8:

\(C\) is said to be complete provided that \(C\) is \(I\)-complete for
each small category \(I\).

Proposition 4.9:

For any category \(C\), the following are equivalent:

(1) \(C\) is complete.

(2) \(C\) has products and equalizers.

Another special case of finite limits is that of a pullback.

Definition 4.10:

Let \(F : A \to C\) and \(g : B \to C\) be two morphisms with the
same codomain. A pullback for the pair \((f, g)\) is a commutative
rectangle

\[
\begin{array}{ccc}
P & \longrightarrow & P \\
\downarrow r & & \downarrow g \\
S & \searrow & C \\
\downarrow f & & \ \\
A & \longrightarrow & C \\
\end{array}
\]

with the following property: if \(U : D \to A\), \(V : D \to B\) are

\(^1\) Herrlich and Strecker, Category Theory, p. 133-134.
morphisms with \( f \circ \mu = g \circ v \), then there is exactly one morphism \( w: D \to P \) with the property \( \mu = s \circ w \) and \( v = r \circ w \). A category is said to have pullbacks if every pair of morphisms in it with the same codomain has a pullback.

**Proposition 4.11:**

Let

\[
\begin{array}{ccc}
P & \xrightarrow{r} & B \\
\downarrow{g} & & \downarrow{g} \\
S & \xrightarrow{r} & \ast \\
\downarrow{f} & & \downarrow{f} \\
A & \xrightarrow{r} & C
\end{array}
\]

be a pullback. If \( f \) is a monomorphism then \( v \) is a monomorphism. If \( f \) is a retraction, then \( r \) is a retraction.

**Proposition 4.12:**

For any category \( \mathcal{C} \), the following are equivalent:

1. \( \mathcal{C} \) is complete.
2. \( \mathcal{C} \) has products and equalizers.
3. \( \mathcal{C} \) has products and pullbacks.

**Proposition 4.13:**

If \( \mathcal{A} \) is a complete category and \( F: \mathcal{A} \to \mathcal{B} \) is a functor, then the following are equivalent:

1. \( F \) preserves limits.
2. \( F \) preserves products and equalizers.
3. \( F \) preserves products and pullbacks.

**Proposition 4.14:**

If the functor \( T: \mathcal{C} \to \mathcal{D} \) preserves pullbacks, then it preserves monomorphisms.
Algebraic Categories:

Theorem 4.15:

If \((A, U)\) is algebraic, then \(U\) preserves and reflects monomorphisms and isomorphisms.

Proof: Since \(U\) is a functor, it preserves isomorphisms: since it has a left adjoint, it preserves limits (and thus preserves monomorphisms) and since it is faithful, it reflects regular epimorphisms so that since a morphism is an isomorphism if and only if it is a regular epimorphism and a monomorphism, \(U\) must reflect isomorphisms.

Proposition 4.16:

Let \((A, U)\) be algebraic and let \(f\) be an \(A\)-morphism. Then

1) \(f\) is an monomorphism if and only if \(U(f)\) is injective.
2) \(f\) is an regular epimorphism if and only if \(U(f)\) is surjective.
3) \(f\) is an isomorphism if and only if \(U(f)\) is bijective.

Proposition 4.17:

Let \((A, U)\) be algebraic \(f: A \rightarrow C\) be an \(A\)-morphism \(g: A \rightarrow B\) be a regular epimorphism in \(A\), and \(h: U(B) \rightarrow U(C)\) be a function such that the triangle

\[ \begin{array}{ccc}
U(A) & \overset{U(f)}{\longrightarrow} & U(C) \\
\downarrow U(g) & & \downarrow h \\
U(B) & \rightarrow & U(C)
\end{array} \]

commutes. Then there exists a unique \(A\)-morphism \(h: B \rightarrow C\) such that \(U(h) = h\).
Proof: Since \( g \) is a regular epimorphism, \((g, B)\) must be the coequalizer of some pair \((p, q)\) of \(A\)-morphisms. Now
\[
U(f \circ p) = U(f) \circ U(p) = h \circ U(g) \circ U(p)
\]
\[
= h \circ U(g) \circ U(q) = U(f) \circ U(q)
\]
\[
= U(f \circ q).
\]
Since \( U \) is faithful, we have \( f \circ p = f \circ q \). Hence, by the definition of coequalizer, there exists a unique morphism \( h: B \to C \) such that \( f = h \circ g \). Since \( U(g) \) is an epimorphism, \( U(h) \circ U(g) = U(f) = h \circ U(g) \) implies that \( U(h) = h \). Uniqueness follows from the fact that \( U \) is faithful.

Proposition 4.18:

Let \((\alpha, \beta)\) be algebraic, \((X, (g_i))\) and \((Y, (m_i))\) be sources in \(A\) such that \((U(Y), (U(m_i)))\) is a mono-source in Set, and \( h: U(X) \to U(Y) \) be a function such that \( U(g_i) = U(m_i) \circ h \) for each \( i \).

Then there exists a unique \(A\)-morphism \( \bar{h}: X \to Y \) such that \( U(h) = h \).

Proof: Let \((U, A)\) be a \(U\)-universal map for \(U(X)\). Then there exists \(A\)-morphisms \( g \) and \( f \) such that the diagram

\[
\begin{array}{ccc}
U(X) & \xrightarrow{U(g)} & U(Y) \\
\downarrow U & & \downarrow U \\
U(X) & \xrightarrow{U(f \circ g)} & U(Y)
\end{array}
\]

Commutes. Thus
\[
U(g_i \circ g) \circ U = U(g_i) \circ U(X) = U(m_i) \circ h = U(m_i \circ f) \circ U,
\]
so that since \( U \)-generates \( A \), it follows that \( g_i \circ g = m_i \circ f \) for each \( i \). Now,
for each \( i \).

\[ U(m_i) \circ h \circ U(g) = U(g_l \circ g) = U(m_l \circ f) = U(m \circ f) \]

so that since \((U(y), (U(m_1)))\) is a mono-source, \( h \circ U(g) = U(f) \). But, \( U(g) \) is a retraction, so that since \( U \) reflects regular epimorphisms \( g \) must be a regular epimorphism. Hence \( f, g, \) and \( h \) satisfy the hypothesis of the previous proposition \#4.17. Consequently, there is a unique \( A \)-morphism \( h \) such that \( U(h) = h \).

Proposition 4.19.

If \((A, U)\) is algebraic, then \( U \) reflects limits.

Proof: Let \( D: I \to A \) be a functor and let \((L, (\xi_i))\) be a source in \( A \) such that \((U(L), (U(\xi_i)))\) is a limit of \( U \circ D \). Since \( U \) is faithful, \((L, (\xi_i))\) must be a natural source for \( D \). If \((X, (\eta_i))\) is some natural source for \( D \), then \((U(X), (U(\eta_i)))\) is a natural source for \( U \circ D \), so that there exists a unique function \( h: U(X) \to U(L) \) such that \( U(X) \to U(L) \) such that \( U(g_i) = U(\xi_i) \circ h \), for each \( i \). Since \((U(L), (U(\xi_i)))\) is a limit, it is a mono-source. Hence, by the previous proposition \#4.18 there is a unique \( A \)-morphism \( \overline{h} \) such that \( U(\overline{h}) = h \), i.e. (since \( U \) is faithful) such that for each \( i, g_i = \xi_i \circ \overline{h} \). Consequently, \((L, (\xi_i))\) is a limit of \( D \).

Theorem 4.20:

If \((A, U)\) is algebraic, then \( A \) is complete and \( U \) preserves and reflects limits.

Proof: Since \( U \) has a left adjoint, it preserves limits, and by the above proposition \#4.19 it also reflects them. Thus, we need only show that \( A \) is complete.
Let $D: I \to A$ be a small functor, let $(L, (\ell_i))$ be a limit of $U \circ D$ and let $(\eta, \lambda)$ be a $U$-universal map for $L$. Then for each $i$ there is a unique morphism $\bar{\ell}_i: A \to D(i)$ such that $U(\bar{\ell}_i) \circ U = \ell_i$. Now for each $I$-morphism $m: i \to j$, the equality

$$U(\bar{\ell}_j) \circ U = \ell_j = (U \circ D)(m) \circ \ell_i =$$

$$(U \circ D)(m) \circ U(\bar{\ell}_i) \circ U = U(D(m) \circ \bar{\ell}_i) \circ U$$

and the fact that $U$ $U$-generates $A$ implies that $\bar{\ell}_j = D(m) \circ \bar{\ell}_i$. \(^1\)

---

\(^1\) Refer to appendix for definitions and proposition that show $U$ $U$-generates $A$. 

Consequently, \((A, (\mathcal{F}_1))\) is a natural source for \(D\), so that 
\((U(A), (U(\mathcal{F}_1)))\) is a natural source for \(U \circ D\). Hence, there exists a unique function \(f: U(A) \to L\) such that for each \(i\), 
\(U(\mathcal{F}_1) = l_i \circ f\). Since \((L, (\mathcal{L}))\) is a limit, it is a mono-source so that the equations 
\[ f \circ f \circ U = U(\mathcal{F}_1) \circ U \circ u = l_i \circ l_i = l_i \circ \mathcal{L} \]
implies that \(f \circ U \circ u = \mathcal{L}\). Thus, \(f\) is a surjection and, as such, is a regular epimorphism. Therefore, \((\mathcal{F}_1, L)\) must be the coequalizer of some pair of functions \(C \to U(A)\). Let \((\mathcal{V}, \mathcal{B})\) be a \(U\)-universal map for \(C\). Then there exist unique morphisms \(r, s:\)
\[ B \to A \] such that 
\[ r = U(r) \circ r \quad \text{and} \quad s = U(s) \circ r. \]
Let \((q, Q) \approx \text{Co eq} \ (r, s)\). Then the equality 
\[ U(q) \circ r = U(q) \circ (U(r) \circ v) = U(q \circ v) \circ v = U(q \circ s) \circ v = U(q) \circ (U(s) \circ v) = U(q) \circ s \]
implies that there is unique function \(h: L \to U(Q)\) with \(U(q) = h \circ f\). Since \(U(q)\) is surjective, \(h\) must be surjective also.

Furthermore, for each \(i\), \(U(e_i \circ r) \circ V = U(\mathcal{F}_1) \circ r = U(\mathcal{F}_1) \circ s \circ V\).

This, together with the fact that \(v\) \(U\)-generates \(B\), implies that 
\[ \mathcal{F}_1 \circ r = \mathcal{F}_1 \circ s. \]
Consequently, for each \(i\) there exists a unique morphism \(q_i: Q \to D(i)\) such that 
\[ \mathcal{F}_i = q_i \circ q_i. \]
Now $U(q_i) \circ h \circ f = U(q_i) \circ U(q) = U(q_i \circ f) = U(\ell_i) = \ell_i \circ f$ so that since $f$ is a surjection, $U(q_i) \circ h = \ell_i$, for each $i$. This provides a factorization the source $(L, (\ell_i))$, where $h$ is an epimorphism.\(^1\) Hence $h$ is an isomorphism. Thus $(U(q), (U(q_i)))$ is a limit of $U \circ D$. Since $U$ reflects limits, then $(Q, (q_i))$ must be a limit of $D$.

Algebraic Functors:

By definition the forgetful functor $U$ of the algebraic category preserves and reflects regular epimorphism as has a left adjoint. There are other properties that distinguishes each algebraic functor from other functors. The algebraic

\(^1\)Definitions are given in the appendix of a epi-sink and mono-factorizations.
functor plays a central role in categorically distinguishing algebra from other areas of mathematics.

Definition 4.21:

A functor is called an algebraic functor if it has a left adjoint and preserves and reflects regular epimorphisms.

Proposition 4.22:

The composition of algebraic functors is algebraic.

Proposition 4.23:

Each algebraic functor is faithful.

Proof:

Suppose that $G: A \to B$ is algebraic and $A \xrightarrow{f} A'$ are $A$-morphisms such that $G(f) = G(g)$. Let $(U, A')$ be a $G$-universal map for $G(A)$. Then there is a unique $A$-morphism $h: A' \to A$ such that the triangle

$$
\begin{array}{ccc}
G(A) & \xrightarrow{u} & G(A') \\
\downarrow{G(A)} & & \downarrow{G(h)} \\
G(A) & \xrightarrow{G(h)} & G(A) \\
\end{array}
$$

commutes.

Hence, since $N G$-generates $A'$ the equality $G(f \circ h) \circ u = G(g \circ h) \circ u$ implies that $f \circ g = g \circ h$. Since $G(h)$ is a retraction, it is a regular epimorphism, so that since $G$ reflects
Each algebraic functor

(1) preserves and reflects monomorphisms;
(2) preserves and reflects isomorphisms;
(3) preserves and reflects \((\text{regular epi, mono})\) factorizations.

Proposition 4.24:

If \(A\) has co-equalizers and if \(U: A \to \text{Set}\), then the following are equivalent:

(1) \((A, U)\) is algebraic category.
(2) \(U\) is an algebraic functor.

This last theorem says that a functor between two algebraic categories is algebraic. This means that one can get a new algebraic functor between other algebraic categories.

Theorem 4.25:

If \((A, U)\) and \((B, V)\) are algebraic categories and \(G: A \to B\) is any functor such that the triangle

\[
\begin{array}{ccc}
A & \xrightarrow{G} & B \\
\downarrow{U} & & \downarrow{V} \\
\text{set} & & \text{set}
\end{array}
\]

commutes,

then \(G\) is algebraic.

Proof: Since \(U\) and \(V\) preserve and reflect regular epimorphisms and limits, \(G\) must do likewise. Thus, we need only show that \(G\)
has a left adjoint, and to do this it is sufficient to show that each B-object, B, externally G-generates at most a set of pairwise non isomorphic A-objects. Let \((\mathfrak{A}, A)\) be a U-universal map for \(V(A)\) and let \(g: B \rightarrow G(\mathfrak{A})\) externally G-generate \(\mathfrak{A}\).

Since \(A\) is regular co-well powered it is sufficient to show that for some morphism \(\widetilde{g}\), \((\widetilde{g}, \mathfrak{A})\) is a regular quotient object of \(A\).

By the definition of universal map, there exists a unique A-morphism \(\overline{g}: A \rightarrow \mathfrak{A}\) such that \(V(\overline{g}) = U(\overline{g}) \circ \mathfrak{u}\). Let \(A \xrightarrow{e} A' \xrightarrow{m} A\) be the (regular epi, mono) - factorization of \(\overline{q}\).

\[
\begin{align*}
V(B) & \xrightarrow{V(g)} U(A) = (V \circ G_1)(A) \\
\end{align*}
\]

By a previous proposition, there exists a unique B-morphism \(h: B \rightarrow G(A')\) such that \(V(h) = (V \circ G)(e) \circ \mathfrak{u}\). Since \(V\) is faithful, this implies that

\[
B \xrightarrow{g} G(\mathfrak{A}) = B \xrightarrow{h} G(A') \xrightarrow{G(m)} G(\mathfrak{A})
\]
Since $g$ externally $G$-generates $A'$, this implies that $m$ is an isomorphism. Hence, $g$ must be a regular epimorphism.
Chapter V

Some Significant Applications of Algebraic Categories in the Mathematical Sciences
A. Finding Equivalent Mathemical Structures Using Algebraic Categories

Mathematicians have been able to establish some interesting results using the concepts of categories. Different mathematical structures have been found to be equivalent in general structure using categorical mappings.

Here the concepts of categorical equivalence are reviewed and extended.

Definition (Review) 5.1

Let \( S: C \rightarrow D \) be a functor. Let \( T: C \rightarrow D \) be a functor. A natural transformation \( \alpha : S \rightarrow T \) is a map which assigns to each object \( A \) of \( C \) an object of \( D \) in such a way that for every morphism \( f: A \rightarrow B \) the diagram is commutative.

\[
\begin{array}{ccc}
S(A) & \rightarrow & T(A) \\
\downarrow \alpha_A & & \downarrow \alpha_T \\
S(f) & \rightarrow & T(f) \\
\downarrow \alpha_{S(f)} & & \downarrow \alpha_{T(f)} \\
S(B) & \rightarrow & T(B)
\end{array}
\]

Also, \( T(f) \circ \alpha_A = \alpha_B \circ S(f) \) for every morphism \( f: A \rightarrow B \) in \( C \).

Definition 5.2 (Review)

A functor \( F: A \rightarrow B \) is called an equivalence if there is another functor \( G: B \rightarrow A \) such that \( F \circ G = 1_B \) and \( G \circ F \cong 1_A \). The equivalence of \( F: A \rightarrow B \) implies that categories \( A \) and \( B \) are equivalent.
The following theorem is used to show equivalence of two categories.

Proposition 5.3

The quadruple \((F, G, \iota_A, \iota_B)\) is an equivalence of the categories \(A\) and \(B\) if

\begin{enumerate}
\item \(F : A \rightarrow B\) is a functor.
\item \(G : B \rightarrow A\) is a functor.
\item \(\iota_A : 1_A \rightarrow G \circ F\) is a natural equivalence.
\item \(\iota_B : 1_B \rightarrow F \circ G\) is a natural equivalence.
\end{enumerate}

Thus \(A\) and \(B\) are said to be equivalent categories.

The purpose of this section of this chapter is to establish the equivalence of a category of dual vector spaces over the same division ring with a bilinear map of a direct product of the vector spaces into the division ring and a second category whose objects are regular linear systems and its morphisms are semilinear transformation of the vector spaces whose conjugate maps the corresponding total subspace into one another.

A vector space is a \(R\) - module with \(R\) as a division ring with unity. Therefore a bilinear map of vector spaces is a bilinear of map of \(R\) - modules. The following definitions and propositions show the first category in an algebraic category.

Proposition 5.4

The following categories are complete and there is for them a canonical choice of limits: \(\text{Sets, Top, Ab, } R\text{-Mod, } Mod_R\text{ and Group.}\)
Definition 5.5

The functor $\hom_a : \mathcal{A}^{\text{op}} \times \mathcal{A} \to \text{Set}$ is called the set-valued hom-functor or morphism functor. The right associated functor with respect to $\hom_a$ and $\mathcal{A}$ or $\hom_a (\mathcal{A}, -)$:

$\mathcal{A} \to \text{Set}$ is called the covariant hom-functor of $\mathcal{A}$ with respect to $\mathcal{A}$.

An example of a hom-functor would be $\text{Hom} : R - \text{mod}^{\text{op}} \to \text{Ring}$ (division ring).

Proposition 5.6

If $\mathcal{A}$ is complete and cocomplete, and if $\mathcal{A}$ is an object then the following are equivalent:

1. $\hom (\mathcal{A}, -)$ is an algebraic functor.
2. $(\mathcal{A}, \hom (\mathcal{A}, -))$ is an algebraic category.

Therefore a category of dual vector spaces over the same division ring with a bilinear map of a direct product of two vector spaces into a division ring is an algebraic category.

Definition 5.7

Let $U$ be a right and $V$ be a left vector space over a division ring $R$. Let $f : V_R \times_R U \to R$ be a bilinear map. Then $(u, v, \theta, f)$ is a pair of dual vector spaces over $R$.

Definition 5.8

The pair $(\theta, h)$ is a semilinear transformation of a left vector space $V$ into $V'$ if $g$ is an isomorphism of $R$ into $R'$ and $h$ is an additive homomorphism of $V$ into $V$ and $(\alpha \cdot v) h = \theta h v + h (\alpha v)$ ($\alpha \in R$, $v \in V$).
where \( \theta \) denotes the value of \( \Theta \) at \( \alpha \). A semilinear transformation of right vector spaces will be denoted \((h, \Theta)\).

**Definition 5.9** Let \((U_R, V, f; f')\) and \((U'_R, V', f'; f')\) be pairs of dual vector spaces. Let \((\Theta, h)\) be semilinear transformation of \(V\) into \(V'\), then the adjoint of \(h\), if it exists, is the function \(d : U'_R \to U_R\) that satisfies
\[
(v, d(u')) = (v, h(u'))',
\]
\((v \in V, u' \in U'_R)\).

**Proposition 5.10**

A semilinear transformation composition is \((\Theta, h) (\Theta', h') = (\Theta \Theta', hh')\).

**Definition 5.11**

\(1_R\) is the identity of the semilinear transformation on \(R^V\).

**Definition 5.12**

Objects \(D\), pairs of dual vector spaces over a division ring, and \(\text{Hom } D\), semilinear transformations with an adjoint, form the category \(D\).

**Definition 5.13**

The conjugate space of \(V\) is denoted \(V^*\) and is the right vector space over \(R\) consisting of all linear form on \(V\).

(\(V^*\) may be denoted \(V^*_R\))
Definition 5.14

\[(V^*, R, V)\] is a pair of dual vector spaces over \(R\) with the bilinear form given by \((v, f) = v_f\).

Definition 5.15

A subspace \(F\) of \(V^*\) is a \(t\) - subspace (total subspace) if for every \(v \in V\), there is a \(f \in F\) such that \(v_f \neq 0\).

Definition 5.16

If \(F\) is a \(t\) - subspace of \(V^*\) for a left vector space \(V_R\), then the dual pair \(F^\perp V_R\) is a regular linear system.

Definition 5.17

\(C\) is a full subcategory of \(D\) by objects \(-C\) which is the set of regular linear systems.

Definition 5.18

For every \(u \in U\), \(f_u\) is defined by \(v f_u = (v, u)\) for \(v \in V\). Let \(f: u \rightarrow f_u\) \((u \in U)\).

The function \(f\) is a linear isomorphism of \(U\) onto a \(t\) - subspace of \(V^*\).

Definition 5.19

The mapping \(f\) is called the natural isomorphism of \(U\) into \(V^*\).

Definition 5.20

The natural image of \(U\) in \(V^*\) under \(f\) is denoted \(\text{nat } U\).

Definition 5.21

The conjugate \(h^*\) of \(h\) is the function on \(R, V^*\) defined as follows:
for every $f' \in V^{*}$, let $h^{*} f'$ be the function on $V$ given by

$$v(h^{*} f) = (v f') \quad (v \in V).$$

**Theorem 5.22**

The function $h^{*} f$ is additive in $V^{*}$

**Proof:**

For any $\alpha \in \mathbb{R}$ and $v \in V$ we obtain

$$\alpha v(h^{*} f') = [(\alpha v) f']^{\Theta^{-1}} = \alpha \left[ (v h f')^{\Theta^{-1}} \right] = \alpha (v h f')^{\Theta^{-1}} = \alpha [v (h^{*} f')]^{\Theta^{-1}},$$

show analogously that $h^{*} f'$ is additive and thus $h^{*} f' \in V$.

Note $(d, \Theta')$ is a semilinear transformation from $\mathcal{U}_{R}^{*} \rightarrow \mathcal{U}_{R}$

and $(u^{*} \alpha') = (d u') \alpha' \Theta^{-1}$

where $(u^{*} \alpha' \in \mathcal{U}_{*}^{*} \alpha' \in \mathbb{R})$.

**Theorem 5.23.**

For any $v \in V$, $f' \in V^{*}$ and $\alpha' \in \mathbb{R}$, then $(h^{*}, \Theta^{-1})$ is a semilinear transformation of $V^{*} \rightarrow V^{*}$ and $h^{*}$ is additive.

**Proof:**

For any $v \in V$, $f' \in V^{*}$ and $\alpha' \in \mathbb{R}$, obtain $v [h^{*} f'] = [v h (f' u')]^{\Theta^{-1}} = [v h f']^{\Theta^{-1}} = (v h f')^{\Theta^{-1}}$. 

which shows that 
\[ h^* f' \alpha' = h^* (L f' \alpha') \]  
Since \( R^* \) is clearly additive, we infer that \( (h^*, \theta^{-1}) : V^* \to V^* \) is a semilinear transformation.

**Theorem 5.24**

Let \( (U_R, \mathcal{R}_R, \mathcal{V}^\prime) \) and \( (U'_R, \mathcal{R}_R', \mathcal{V}^\prime') \) be pairs of dual vector spaces, and let \( (\Theta, \mathcal{R}) : \mathcal{V} \to \mathcal{V}^\prime \) be a semilinear transformation. Then \( h \) has an adjoint if and only if \( h^* \) \( (\text{nat } U) \subseteq \text{nat } U \).

**Proof:**

Let \( d \) be the adjoint of \( h \) and let \( w \in U' \). Then for any \( v \in V \) we obtain 
\[ v(h^* f_w) = (vh f_w)^{\Theta^{-1}} = (vd f_w)^{\Theta^{-1}} = (v, d w') = v f_d w' \]
so that \( h^* f_w = f_d w' \) \( \text{nat } U \). Consequently \( h^* (\text{nat } U) \subseteq \text{Nat } U \).

Conversely, assume that \( h^* \) \( (\text{nat } U) \subseteq \text{nat } U \). Define a function \( d \) on \( U \) by: \( d w' = w \) if \( h^* f_w \) \( (w \in U') \). Reversing the previous calculation, we see that \( (d, \Theta^{-1}) \) is adjoint \( (\Theta, \mathcal{R}) \).

**Proposition 5.25**

Let \( (F, R, V) \) and \( (F', \mathcal{R}_1, \mathcal{V}^\prime) \) be regular linear systems and let \( (\Theta, \mathcal{R}) : \mathcal{V} \to \mathcal{V}^\prime \) be a semilinear transformation. Then \( h \) has an adjoint if and only if \( h F' \subseteq F \).

**Theorem 5.26**

The quadruple \( (W, \mathcal{L}_i, \mathcal{I}_i) \) is an equivalence of the categories \( D \) and \( C \).
Proof:

$W$ is a functor on $D$ defined by $W(\mathbf{u}_R, R, V; f) = \operatorname{Nat}(\mathbf{u}_R, R, V; f)$ where $(\mathbf{u}_R, R, V; f) \in \text{ob } D$ and $W(\mathbf{u}, W) = (\mathbf{u}, h)$ for $(\mathbf{u}, h) \in \text{Hom } D$. $W$ is the functor $W : D \rightarrow C$.

$1_C : C \rightarrow D$ is the functor between the categories going in the opposite direction. Thus $W(1_C) = 1_C$. For every $(\mathbf{u}_R, R, U; f) \in \text{ob } D$, we let $(\mathbf{u}_R, R, V; f) = (t_R, t, \mathbf{v})$. From the proposition it follows that is a natural equivalence of the functors $1_D$ and $1_W$. For every $(F, R, V) \in \text{Ob } C$, let $t(F, R, V) = (t_R, t_V)$. Thus the quadruple $(\mathbf{u}, 1_C, i, t)$ is an equivalence of the categories $D$ and $C$.

B. Theorems in One Area Proved Using Algebraic Categories

The purpose of this section is to present theorem in other areas of mathematics whose proofs involve algebraic category theory and/or concepts.

Category of Distributive Lattices

The categorical concept of projections is used to prove a property about lattices.

Definition 5.27

Let $K$ be a category of algebraic. An object $P$ of $K$ is called projective, if, for any onto map $\alpha : A \rightarrow B$, and any map $B : P \rightarrow B$, there is a map $\lambda : P \rightarrow A$ with $\lambda \circ \alpha = B$. 


Proposition 5:28

Let $F$ be a free algebra in $K$. Then $F$ is projective.

Proposition 5:29

Let $F$ be a free algebra in $K$ and the $G$ be a retract of $F$. Then $G$ is projective. Definition 2 an element of a distributive lattice is meet-irreducible if $a = b \land c$ implies that $a = b$ or $a = c$.

Proposition 5:30

If free algebras exist on any set of generators, then only retracts of free algebras are projective.

Theorem 5:31

A finite distributive lattice $P$ is projective in $D$ if and only if the join of any two meet-irreducible elements is meet irreducible. ($D$ is a category of distributive lattices)

Proof: 1 Let $P$ be projective

Previous Proposition. By Lemma 10 we can assume that $P \leq F$, where $F$ is free on $x_i, i \in I$ and $P$ is a retraction: $F \rightarrow P$. Let $a$ and $b$ meet irreducible and let $a \lor b$ be meet reducible - that is for some $C, d \in P$, $a \lor b \geq c \land d$, $a \lor b \geq c, a \lor b \neq d$. Let $C = \lor(\land c_k / k \in K)$

---

1 Lattice Theory first concepts and Distributive Lattices by Gratzer
and let \( d = \bigvee (\bigwedge_{j \in E} c_j) \), where \( c_k \) and \( D_k \) are finite subsets of \( \{ x_i \mid i \in I \} \).

Then \( C = c_P = \bigvee (\bigwedge_{k \in K} c_k) \),
\[ d = \bigvee P = \bigvee (\bigwedge_{j \in E} c_j). \]

Therefore, \( K \) and \( \bigvee P \) exist such that \( \bigwedge_{j \in E} c_j \) and \( \bigwedge_{j \in E} c_j \) at the same time, \( \bigwedge_{j \in E} c_j \leq \bigwedge_{j \in E} c_j \) and \( \bigwedge_{j \in E} c_j \leq \bigwedge_{j \in E} c_j \) therefore,
\[ \bigwedge_{j \in E} c_j \leq a \text{ or } \bigwedge_{j \in E} c_j \leq b. \]

by theorem: (Let \( L \) be a distributive lattice generated by \( \{ a_i \mid i \in I \} \)
\( L \) is freely generated by the \( a_i \) if and only if the validity in \( L \)
the validity of a relation of the form \( \bigwedge \{ \bigvee (a_i \mid i \in I) \} \leq \bigvee (a_i \mid i \in I) \)
implies that \( I_0 \wedge I_1 \neq \emptyset \).)

Applying \( P \) we get \( \bigwedge_{j \in E} c_j \leq a \) or \( \bigwedge_{j \in E} c_j \leq b \), which
means that either \( a \) or \( b \) is meet - reducible, a contradiction.

Conversely, let \( p \) satisfy the condition of the theorem and let
\( m_i, i \in I \) be the meet - irreducible elements of \( P \). Let \( F \) be
the free distributive lattice on \( x_i, i \in I \) and let \( \alpha \) be the
homomorphism of \( F \) onto \( P \) extending \( x_i \to m_i, i \in I \).
Let \( G \) be the join - subsemilattice of \( F \) generated by \( x_i, i \in I \)
how we define a map \( \beta : P \to F \) by \( \alpha \beta = \bigwedge (x_1 x \in G, x \geq a) \wedge \beta \).
In other words, \( \alpha \beta \) is the smallest element in the subset \( \alpha \beta \)
of \( F \). Distributivity shows that
\( (a \vee b) \beta = a \beta \vee b \beta \)
since any \( x \) in \( G \) such that \( x < a \) \( x \leq a \) is of the form \( x = y \wedge z \)
\( y \in G, y \vee z \leq a, z \in G, z x \geq b \) and conversely. To show
that preserves meets, note that, by the assumptions of the theorem,
is meet - irreducible for all \( X \) in \( G \). Thus \( x \) is
equivalent to the condition $xq \geq a$ or $xa \geq b$. This proves that
is a homomorphism of $P$ into $F$.

Finally, we have $a = \Lambda (m_i | m_i \geq a)$ for every element $a$ of $P$. Consequently, $qM = a$; therefore, $P$ is a retract of $F$ and is
thus projective by proposition 5.30

Category Theory in Algebraic Topology

Presented are three theorems from G. H. Verdier paper "On a
Theorem of Wilder in Applications of Categorical Algebra published
by the American Mathematical Society, pages 184–191. The theorems
are present to show how much ideas from algebraic categories are
used to prove theorems in algebraic topology. They are presented
without definitions or explanation.

Theorem 5.32

Let $X$ be a connected, locally simply connected, finite
dimensional compact space. Then, for any perfect complex $F$ on $X$, we have: $W_A (X, F)* = W_A (X, D F)$. 

Proof: Sketch of the proof Lep $P: \tilde{X} \longrightarrow X$ be the universal
covering of $X$. By a previous proposition we may assume that $F$ is a
bounded complex of $c$-soft sheaves. Let us consider the complex

$D_X, A[\pi, (\ neut)] P! P^* F$ Viewed as an $A^o [\pi, (\ neut)]$ complex (via the
isomorphism $A[\pi, (\ neut)] \longrightarrow A[\pi, (\ neut)]^o$).

Using $U \longrightarrow Hom_A (\Gamma_C (U, S(F)), \mathcal{I} (A))$ and
definition of $p!$, we see that the complexes
have isomorphic injective resolutions. Moreover, since \( p \) is a covering, we have a canonical isomorphism \( P \cong A P^* \cong P^* D X, A F \).

Therefore complexes \( \Gamma(X, P ! P^* D X, A F) \) and
\[
\Gamma X, D_X, A[\pi_1(X)] p ! P^* F
\]
have isomorphic injective resolutions.

Let \( \Gamma A[\pi_1(X)] \) be a resolution of \( A[\pi_1(X)] \) by \( A[\pi_1(X)] \) bimodules injective on the right. It follows from the Poincaré duality theorem that the complexes \( \Gamma(X, D X, A[\pi_1(X)] p ! P^* F) \) have isomorphic injective resolutions. These exist bounded complex of finitely generated projective \( A[\pi_1(X)] \) - modules \( C(X, F) \) and a resolution \( C(X, F) \to \Gamma(X, P ! P^* F) \). We have therefore two maps of complexes
\[
\text{Hom} A[\pi_1(X)] \left( C(X, F), A[\pi_1(X)] \right) \\
\text{Hom} A[\pi_1(X)] \left( \Gamma(X, P ! P^* F), I(A[\pi_1(X)]) \right)
\]
which induce isomorphism on the cohomology and the proposition follows.

Theorem 5.33

Let \( X \) be connected locally simply connected finite dimensional compact space, \( F \) a perfect complex of sheaves of \( A \) - modules, \( M \) a locally constant sheaf of finitely generated and projective \( A \) - modules. Then
\[
W_A(X, M \otimes A F) = c l (M) \cdot W_A(X, F).
\]
Proof:

By a previous proposition we may assume that $F$ is a bounded complex of $\mathcal{C}$-soft $A$-modules. Let $p : \tilde{X} \rightarrow X$ be the universal covering of $X$. The $A[[\pi_1(\tilde{X})]]$-complex $\Gamma^c_c(\tilde{X}, p^*F)$ has a resolution $\mathcal{C}(X, F)$ by a bounded complex of finitely generated and projective $A[[\pi_1(\tilde{X})]]$-modules.

Since we have a canonical isomorphism $\Gamma^c_c(\tilde{X}, p^*M \otimes_A F) \cong \Gamma^c_c(\mathcal{X} \times F, M \otimes_A M_X)$, where $M$ is any stalk of $M$ with its natural action of $\pi_1(\tilde{X})$, the complex $C(X, F) \otimes_A M_X$ is a resolution of the complex $\Gamma^c_c(\tilde{X}, p^*M \otimes_A F)$.

Corollary 5.34

Let $X$ be a compact connected $n$-dimensional topological variety with boundary. Let $dX_j$, $1 \leq j \leq g$ be the different connected components of its boundary, $i_j : dX_j \rightarrow X$ the inclusion maps, and let $\Lambda_{dX}$ be the orientation $\mathbb{Z}[[\pi_1(\tilde{X})]]$-module. Then

$$(-1)^{n-1} \omega_2(X, \mathbb{Z}) + (-1)^n \sum_j i_j \ast (\omega_2(dX_j, \mathbb{Z})) = c_2(\Lambda_{dX}). \omega_2(X, \mathbb{Z}).$$

Proof: We have

$$\omega_2(X, \mathbb{Z}) = \omega_2(X, T_X)$$

where $T_X$ is the orientation complex of $X$. The complex $T_X$ has only one zero cohomology sheaf $Q_X$ in dimension $-N$. Hence

$$\omega_2(X, \mathbb{Z}) = -1^n \omega_2(X, \mathcal{O}_X).$$

The sheaf $Q_X$ is locally constant free rank one on $X - dX$ and its restriction to $dX$ is zero.
Let $j: X - \partial X \to X$ be the inclusion map. We have an exact sequence $0 \to O_X \to j_* O_X \to j_* O_X / \partial X \to 0$.

Hence $\gamma_2 (X, O_X) = \gamma_2 (X, j_* O_X) + \gamma_2 (X, j_* O_X / \partial X)$. The sheaf $j_* O_X$ is locally free of rank one and is defined by the orientation module $\Lambda_X$. The formula follows from previous proposition and trivial manipulations.
Bibliography


