An extension of Cauchy’s integral formula

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AN EXTENSION OF CAUCHY'S INTEGRAL FORMULA

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. CAUCHY INTEGRAL AND CAUCHY-TYPE INTEGRALS</td>
<td>2</td>
</tr>
<tr>
<td>III. EXISTENCE OF DERIVATIVES OF HIGHER ORDER OF A REGULAR FUNCTION</td>
<td>3</td>
</tr>
<tr>
<td>IV. THE BOUNDARY VALUES OF A CAUCHY TYPE INTEGRAL</td>
<td>5</td>
</tr>
<tr>
<td>Particular Cauchy-type Integral</td>
<td>5</td>
</tr>
<tr>
<td>General Cauchy-type Integral</td>
<td>9</td>
</tr>
</tbody>
</table>
CHAPTER I
INTRODUCTION

The main purpose of this thesis is to extend the fundamental formula of Cauchy's Integral (i.e., in mathematical form

\[ \int_T \frac{\varphi(z)}{z} \, dz = F(z) \]

where, T is any arbitrary rectifiable Jordan's curve lying in a given region G where \( \varphi(z) \) is regular) to more generalized case in which \( \varphi(z) \) may be irregular (but continuous) in a region G. Moreover, it has been found that at each point on the path T, there are two boundary values. For each of these boundary values, a formula is established. Before reaching this purpose, I have introduced that all derivatives of higher orders of a regular function in a given region exist.
CHAPTER II
CAUCHY INTEGRAL AND CAUCHY-TYPE INTEGRALS

In the theory of complex functions, the so-called Cauchy-type integrals play an important role. The integral

$$\frac{1}{2\pi i} \int_T \frac{\varphi(\xi)}{\xi - z} \, d\xi$$

is called a Cauchy-type integral. Here $T$ is an arbitrary rectifiable path, $\varphi(\xi)$ is a continuous function on $T$, $z$ is any point not on $T$. Moreover, if the two additional conditions

1. $T$ is a closed Jordan curve,
2. $\varphi(z)$ is regular within $T$.

are satisfied, then the formula becomes Cauchy integral.

For instance, the following integrals are Cauchy-type integrals

(a) $$\frac{1}{2\pi i} \int_T \frac{f(z)}{x - z} \, dx$$

where $f(x) \neq 0$ is a continuous function on the path---an interval of real axis. If $f(x) = 1$, and $T = [-1, 1]$, we have

$$\frac{1}{2\pi i} \int_{-1}^{+1} \frac{dx}{x - z}$$

(b) $$\frac{1}{2\pi i} \int_{|z| = 1} \frac{\xi \, d\xi}{1 - z}$$

The reason for (b) not being a Cauchy integral is that $f(z) = \overline{z}$ is not regular within the unit circle. This can be easily proved by using the Cauchy-Riemann differential equations.

(c) $$\frac{1}{2\pi i} \int_{|\xi| = 1} \frac{\xi \, d\xi}{\xi - z}$$

Since $f(z) = 1/z$ has a simple pole at the origin, it is not a Cauchy integral.
CHAPTER III
EXISTENCE OF DERIVATIVES OF HIGHER ORDER
OF A REGULAR FUNCTION

Obviously, for every subregion $g$ of a given region $G$ without intersecting $T$—a path lying in $G$, the Cauchy-type integral defines a single-valued function $F(z)$. We are going to prove that this function possesses all derivatives of all orders. Moreover, its $n$th derivative can be obtained by differentiating the integrand $n$-times with respect to $z$, i.e.

\[ F^{(n)}(z) = n!/2\pi i \int_T \frac{\varphi(\xi) d\xi}{(\xi - z)^{n+1}} \]

We prove this by the finite induction method. By definition, this is true for $n=0$. Assume for $n=k$ the formula (1) holds. Now we prove it also holds for $n=k+1$,

\[ \lim_{z' \to z} \left\{ \frac{F^{(n)}(z') - F^{(n)}(z)}{z' - z} \right\} \]

Choose any closed circle $K: |z' - z| < \rho$ within $G$. Let $\delta > 0$ be the distance between the circumference and the contour $T$. Secondly let $K: |z| < R$ be a circle with the origin as a center, such that it is large enough to enclose $K$ and $T$. For any point $z \in K$, we have

\[ F^{(n)}(z') - F^{(n)}(z) = \frac{n!}{2\pi i} \int_T \varphi(\xi) \frac{(\xi - z)^{n+1}}{(\xi - z')^{n+1}} d\xi \]

or, assume $\xi - z = t$, $z' - z = h$, hence $\xi = z + h$;

\[ F^{(n)}(z+h) - F^{(n)}(z) = \frac{n!}{2\pi i} \int_T \varphi(\xi) \frac{(t-h)^n t(t-h)^{n-1} \cdots t^n d\xi}{t^{n+1}(t-h)^{n+1}} \]

We are going to prove the expression (4) approaches the following limit

\[ \Psi(z) = \frac{(n+1)!}{2\pi i} \int_T \frac{\varphi(\xi) d\xi}{(\xi - z)^{n+2}} = \frac{(n+1)!}{2\pi i} \int_T \frac{\varphi(\xi) d\xi}{(\xi - z)^{n+2}} \]
Consider the difference:

\[
\frac{F^{(n)}(z+h) - F^{(n)}(z)}{h} - \frac{\psi(z)}{2\pi i} \int_T \varphi(\xi) \frac{t-h}{(t-h)^{n+1}} \cdot \frac{d\xi}{(\xi - z)^{n+1}} = n! \int_T \varphi(\xi) \frac{t-h}{(t-h)^{n+1}} \cdot \frac{d\xi}{(\xi - z)^{n+1}}
\]

\[
\frac{t}{(t-h)^{n+1}} \cdot \frac{d\xi}{(\xi - z)^{n+1}} = n! \frac{h}{2\pi i} \int_T \varphi(\xi) \frac{t}{(t-h)^{n+1}} \cdot \frac{d\xi}{(\xi - z)^{n+1}}
\]

under our conditions:

\[
2R > |t| = |\xi - z| > \delta, \quad 2R > |t-h| = |\xi - z'| > \delta',
\]

Secondly, let \( \mu = \max_{T} |\varphi(\xi)| \), \(\lambda\) is the length of \( T \). From (5), we get

\[
\left| \frac{F^{(n)}(z+h) - F^{(n)}(z) - \psi(z)}{h} \right| = n! \frac{h}{2\pi} \frac{\mu}{\lambda} \frac{(2R)^{n+2}(2R)^{n+3}}{\delta^{2n+3}}
\]

\[
\lim_{h \to 0} \frac{F^{(n)}(z+h) - F^{(n)}(z)}{h} = F^{(n)}(z) = \psi(z) = \frac{(n+1)!}{2\pi \lambda}
\]

\[
\int_T \frac{\varphi(\xi) d\xi}{(\xi - t)}
\]
CHAPTER IV
THE BOUNDARY VALUES OF A CAUCHY TYPE-INTEGRAL

Let

\[ F(z) = \frac{1}{2\pi i} \int \frac{f(\xi) \, d\xi}{\xi - z} \]

be a Cauchy integral. Therefore in the region g i.e. within the closed contour \( L \), \( f(z) \) and \( F(z) \) are identical, whereas \( f(z) \) is a regular function within \( L_0 = G \). Outside the contour, the value of the integral is equal to zero. Hence if \( z_0 \) is any point on \( L \), when \( z \) approaches from inside, the integral becomes zero.

Therefore, for each point on \( L \), Cauchy integral has two boundary values; namely inside the \( L \), \( f(z) \) and zero outside the \( L \).

Particular Cauchy-type Integral. Now we want to consider the Cauchy-type integral

\[ \frac{1}{2\pi i} \int \frac{\varphi(\xi) \, d\xi}{\xi - z} \]

under the following conditions:

1. \( r \) is a rectifiable Jordan curve,
2. \( \varphi(\xi) \) is regular in the neighborhood of every point on \( r \). Under these conditions, for every point \( z \in r \) other than the endpoints of \( r \) the integral has two boundary values. In this case it differs from the case of Cauchy integral in that the boundary values cannot be directly expressed as \( \varphi(\xi_0) \). However their difference still can be expressed as \( \varphi(\xi_0) \), if one value is selected as subtrahend and the other as minuend. Hence we find the same rule as in Cauchy integral i.e. from inside to outside, it has

\[ f(\xi_0) - 0 = f(\xi_0) \]

As for the proof of the above discussion, we take a neighborhood \( U \) of point \( \xi_0 \) such that for every point in \( U \) the function \( \varphi(z) \) is regular. Take \( \xi_0 \) as center, \( r \) the radius, such that \( T: \mid z - \xi_0 \mid = r \), \( T \subseteq U \).

As figure 1 shows,
choose \( r \) small enough so that some points after \( \xi_o \) and before \( \xi_o \) meet \( T \) also (say \( D, E, \) and \( C, A \)). Tracing the curve \( r \) from \( \xi_o \) forward, the curve meets \( T \) at \( B \). From \( \xi_o \) backward, it meets \( T \) at \( A \). Points \( A \) and \( B \) bisect circle \( T \) into two arcs, namely \( a-a \) and \( a-BB \). These two arcs combine with arc \( AB \) to form two closed Jordan curves, namely \( r_1(ABBA) \) and \( r_2(ABBA) \). Their enclosed subregions \( g_1 \) and \( g_2 \) also lie within \( T \).

We note that in our assigned notations, subregion \( g_1 \) lies on the left side of an observer who traces along \( r \) in arrow direction as shown in the figure while the subregion \( g_2 \) lies on the right side of the observer. Now let us suppose \( z \in g_1 \) approaches \( \xi_o \). Since \( z \) lies outside contour \( r \) which together with subregion \( g_2 \) also belongs to \( U \). \( \varphi(\xi) \) is regular in \( U \). From Cauchy integral, we have

\[
(4.1) \quad \frac{1}{2\pi i} \int_{r} \frac{\varphi(\xi)}{\xi-z} d\xi = 0
\]

or

\[
\frac{1}{2\pi i} \int_{AB} \frac{\varphi(\xi)}{\xi-z} d\xi = \frac{1}{2\pi i} \int_{ABB} \frac{\varphi(\xi)}{\xi-z} d\xi
\]

Hence, when \( z \in \mathcal{g}_2 \), replace \( AB \), for circular arc \( AbB \) and integrate. Then the value of the integral

\[
(4.2) \quad F(z) = \frac{1}{2\pi i} \int_{r} \frac{\varphi(\xi)}{\xi-z} d\xi
\]
is unchanged. Therefore we can rewrite the integral as follows:

\[ F(z) = \frac{1}{2\pi i} \int_{r' + AB - z} \frac{\varphi(z) dz}{z - \xi} \]  

Here \( r' \) is the remaining part of \( r \), when \( AB \) arc is cut away.

However, \( r' + AB \) is rectifiable curve (possibly not Jordan's curve).
Hence the integral (4.3) is a Cauchy-type integral. As what we have shown in chapter II, this integral represents a regular function \( \Phi_1(z) \), which is regular in any arbitrary neighborhood of \( \xi_0 \) without containing any point of \( r' + AB \). Moreover, for any point in \( \xi_1 \)

\[ \lim_{z \to \xi_0} F(z) = \lim_{z \to \xi_0} \Phi_1(z) = \Phi_1(\xi_0) = \frac{1}{2\pi i} \int_{r' + AB - z} \frac{\varphi(z) dz}{z - \xi} \]

Similarly, for the same reason as above, we have

\[ F(z) = \frac{1}{2\pi i} \int_{r' + AB - z} \frac{\varphi(z) dz}{z - \xi} \]

The integral on the right-side represents an analytic function \( \Phi_2 \),
for any point in \( \xi_2 \), \( \Phi_2 \) and \( F(z) \) are identical, i.e.,

\[ \lim_{z \to \xi_0} F(z) = \lim_{z \to \xi_0} \Phi_2(z) = \Phi_2(\xi_0) = \frac{1}{2\pi i} \int_{r' + AB - z} \frac{\varphi(z) dz}{z - \xi} \]

So far, we have found the two boundary values of the Cauchy-type integral at an arbitrary point on the rectifiable curve \( r \), namely (4.4) and (4.6). The one corresponding to \( z \in \xi_2 \), approaches \( \xi_0 \) as a limit and may be denoted as the Left Side Boundary Value (see the figure) and the other one is denoted as Right Side Boundary Value .
We denote them as \( F_1(\xi_0) \) and \( F_2(\xi_0) \) respectively.

\[ F_1(\xi_0) - F_2(\xi_0) = \frac{1}{2\pi i} \int_{\text{AbD - AaB}} \frac{\varphi(z) dz}{z - \xi} = \frac{1}{2\pi i} \int_{\text{C}} \frac{\varphi(z) dz}{z - \xi} \]
From our hypothesis, this integral is nothing new but a Cauchy integral obtained by integrating $\varphi(z)$ along the circle $T$. Therefore,

\[ F_1(z) - F_2(z) = \varphi(z) \]

This is our conclusion.

Example.—Consider the Cauchy-type integral

\[ F(z) = \frac{1}{2\pi i} \int_{\Delta} \frac{dx}{x - z} \]

where $\Delta$ is the closed interval on the real axis $-1 \leq x \leq 1$ (integrating along the positive direction of $x$-axis). For simplicity, we take $\Delta = 0$. Here $\varphi(z) = 1$ is analytic in the whole complex plane. Accordingly take the origin as center, construct any arbitrary circle. (The only condition we have to consider is that such a circle must intersect the closed interval at certain points.) Let us take $T$ as unit circle. Hence the arc $AB$ is identical with the interval $\Delta$ and $\Delta B$, $\Delta A$ corresponds the upper and lower half circles respectively.

From (4.7), we have

\[ F_1(0) - F_2(0) = \varphi(0) = 1. \]

Observe for each boundary values in turn, we have, by (4.4)

\[ F_1(0) = \frac{1}{2\pi i} \int_{\Delta} \frac{1}{x} dx = \frac{1}{2\pi i} \int_{\Delta} \frac{e^{i\theta} d\theta}{e^{i\theta}} = \frac{1}{2} \]

and by (4.6)

\[ F_2(0) = \frac{1}{2\pi i} \int_{\Delta} \frac{1}{x} dx = \frac{1}{2\pi i} \int_{\Delta} \frac{e^{i\theta} d\theta}{e^{i\theta}} = -\frac{1}{2} \]
General Cauchy-type Integral. — For the result on (4.8), we want to give another important suggestion. In the last statement, we assume \( \varphi(\xi) \) is regular but actually we often meet the case that \( \varphi(\xi) \) is not regular in application.

Hence for the same result, we have to give another proof.

As before, \( r \) is a Jordan rectifiable curve and \( \varphi(\xi) \) is continuous function on \( r \). Take \( r \) as a function of the length of the arc \( s \), where \( s \) is calculated from the beginning point.

\[
\xi = \lambda(s), \quad 0 \leq s \leq 1
\]

Here 1 is the length of \( r \). Also assume \( \xi_0 = \lambda(s_0) \) and \( s_0 > 0, \lambda(s_0) \neq 0 \).

Secondly, we assume that there exist two positive numbers \( K \) and \( \alpha \), such that

\[
(4.10) \quad |\varphi(\xi) - \varphi(\xi)| < K|\xi - \xi_0|^\alpha
\]

Under these assumptions, we can easily see that, the integral

\[
(4.11) \quad I_0 = \frac{1}{2\pi i} \int_r \frac{\varphi(\xi) - \varphi(\xi_0)}{\xi - \xi_0} \, d\xi
\]

is absolutely convergent.

For the proof of (4.11) we have to restrict the value of \( s \), such that, \( a \leq s \leq b \) (which is denoted by \([a,b])\).

For any arbitrary section of \([a,b]\) of the arc which does not contain \( \xi_0 \). (For definiteness, say \( s_0 < a < b \)).

From (4.10) we get

\[
\left| \frac{1}{2\pi i} \int_{[a,b]} \frac{\varphi(\xi) - \varphi(\xi_0)}{\xi - \xi_0} \, d\xi \right| < \frac{K}{2\pi} \int_{[a,b]} |\xi - \xi_0|^\alpha \, ds
\]

From assumption,

\[
\lim_{s \to s_0} |\xi - \xi_0| = |\lambda'(s)| \neq 0,
\]

and we observe that, when \( |\xi - \xi_0| < \rho \)

\[
\left| \frac{\xi - \xi_0}{s - s_0} \right| > \frac{1}{2} |\lambda'(s)| > 0;
\]
when \(|\frac{3}{2} - \frac{s'}{2}\)| > 0

\[
\left| \frac{1}{2\pi i} \int_{[a,b]} \frac{\Phi(i) - \Phi(i)}{i - s} \, ds \right| < K^b_a (s - s_0)^{d-1}
\]

\[
= K' \frac{(b-s_0)^d - (a-s_0)^d}{d} \to 0.
\]

Therefore, integral (4.11) is absolutely convergent.

Let us turn back to the Cauchy-type integral

\[
F(z) = \frac{1}{2\pi i} \int \frac{\frac{\Phi(i)}{i}}{i - z} \, dz.
\]

Express it as follows:

\[(4.12)\]

\[
F(z) = \frac{1}{2\pi i} \int \frac{\frac{\Phi(i)}{i} - \frac{\Phi(i_0)}{i_0}}{i - z} \, dz + \frac{1}{2\pi i} \int \frac{\frac{\Phi(i_0)}{i_0}}{i - z} \, dz.
\]

Want to prove

\[(4.13)\]

\[
\lim_{z \to z_0} \frac{1}{2\pi i} \int \frac{\frac{\Phi(i)}{i} - \frac{\Phi(i_0)}{i_0}}{i - z} \, dz = I_0.
\]

We assume \(z\) approaches \(z_0\) within the area \(g_o\), where \(g_o\) is an arbitrary angular area with \(z_0\) as vertex and angle-bisection identical with the normal line at \(z_0\). The angle subtended is less than \(\pi/2\).

This manner of \(z\) approaching \(z_0\) is known as" approaching \(z_0\) along non-tangential direction." For example, \(z\) approaches along the normal line. It is easy to see that there is a suitable neighborhood of \(z_0\), such that the neighborhood does not contain any point of \(r\). For if it is not the case, there will be a sequence of points \(z_n = \lambda(s_0) \in r\) which lie in \(g_o\) and approach \(z_0\) as a limit. We may also consider \(\{s_n\}\) convergent, i.e. \(\lim s_n = s'\), so that \(\lim z_n = \lambda(s')\)

But on the other hand,

\[
\lim_{n \to \infty} z_n = z_0 = \lambda(s_0);
\]
Since \( r \) is a Jordan curve (and different from its endpoint), the conclusion may be drawn that \( s' = s_o \). Hence vector \( \vec{z}_n - \vec{z}_o \) must approach the tangent line at \( \vec{z}_o \). There cannot be a point lying on \( g \theta_o \).

Let \( \theta \) be such that \( 0 < \theta < \frac{\pi}{2} \). Consider a neighborhood \( |z - z_0| < \rho \) such that there is no point \( z \in r \) lying within \( g \theta_o \). Let \( g \theta_p \) be the region in question. Prolong \( r \) in both directions to meet the circle \( |z - z_0| = \rho \) at points \( A \) and \( B \), we obtain

\[ \sigma_p = [s - \epsilon', s + \epsilon''] \subset r. \]

Obviously, for any arbitrary point \( z \in g \theta_p \) and \( \vec{z} \in \sigma_p \), we have

\[ \frac{|z - \vec{z}_o|}{|z - \vec{z}|} < \csc(\theta_o - \theta). \]

\[ \text{Fig. 2} \]

Hence

\[
A = \frac{1}{2\pi i} \int_{x} \frac{\Phi(x) - \Phi(x_o)}{\overline{x - z}} \, \overline{d\vec{z}}
\]

\[ = \left| \frac{1}{2\pi i} \int_{x} [\Phi(x) - \Phi(x_o)] \frac{I_o - z}{(\overline{x - z})(\overline{\vec{z} - \vec{z}_o})} \, d\vec{z} \right| \leq \]

\[ \leq \left| \frac{1}{2\pi i} \int_{x - \sigma_p} [\Phi(x) - \Phi(x_o)] \frac{I_o - z}{(\overline{x - z})(\overline{\vec{z} - \vec{z}_o})} \, d\vec{z} \right| + \csc(\theta_o - \theta) \frac{1}{2\pi} \int_{\sigma_p} \frac{|\Phi(x) - \Phi(x_o)|}{|\vec{z} - \vec{z}_o|} \, ds \]
Given \( \varepsilon > 0 \) arbitrarily, choose \( \rho \) sufficiently small so that \( i \leq \rho / 2 \).

Fix \( \rho \), then we denote the distance between \( \bar{z}_0 \) and \( r - \sigma \) by \( \delta \), \( \delta > 0 \); hence if \( \bar{z} \in r - \delta \), then \( |\bar{z} - \bar{z}_0| \geq \delta > 0 \), i.e. \( |\bar{z} - z| \geq \delta - |\bar{z}_0 - z| \).

Hence when \( |z - \bar{z}_0| < \rho/2 \), we have

\[
i < \frac{1}{2\pi} \int_{r - \delta} \left| \frac{\varphi(\bar{z}) - \varphi(r)}{\bar{z} - r} \right| ds \leq \frac{|r - z|}{\pi \delta^2} \int_{r} \left| \varphi(\bar{z}) - \varphi(r) \right| ds.
\]

Therefore, for all \( z \) sufficiently close to \( \bar{z}_0 \), we have \( i \leq \rho/2 \). Hence for all \( z \) , sufficiently close to \( \bar{z}_0 \) and belong to \( r - \delta \), we have \( A \leq \rho \), i.e. We have proved the relation (4.13) exists, when \( z \rightarrow \bar{z}_0 \) along the non-tangential direction.

Now consider the second integral of (4.12). Obviously it is a Cauchy integral with constant function. Hence it is also a Cauchy-type integral. \( \varphi(z) = \varphi(\bar{z}_0) \) and its value is

\[
\ln \frac{Q - z}{P - z},
\]

where \( P \) is the starting point and \( Q \) is the ending point. But we do not want to make use of this explanation. Using the result from above, namely (4.4) and (4.6), we get

\[
\begin{align*}
\varphi_1(\bar{z}_0) &= \frac{1}{2\pi i} \int_{r^1 - \delta_1} \frac{\varphi(\bar{z}_0)}{\bar{z} - \bar{z}_0} d\bar{z}, \\
\varphi_1(\bar{z}_0) &= \frac{1}{2\pi i} \int_{r^1 - \delta_1} \frac{\varphi(\bar{z}_0)}{\bar{z} - \bar{z}_0} d\bar{z}.
\end{align*}
\]

Comparing with (4.12), (4.13) and (4.14), we draw a conclusion that under the hypothesis of the statement about \( r, \varphi(\bar{z}) \) and \( \varphi_1(\bar{z}_0) \), the two boundary values of the Cauchy-type integral, \( \varphi_1(\bar{z}_0) \) and \( \varphi_1(\bar{z}_0) \), exist. They can be expressed as:

\[
\begin{align*}
\varphi_1(\bar{z}_0) &= \frac{1}{2\pi i} \int_{r} \frac{\varphi(\bar{z}) - \varphi(\bar{z}_0)}{\bar{z} - \bar{z}_0} d\bar{z} + \frac{1}{2\pi i} \int_{r^1 - \delta_1} \frac{\varphi(\bar{z}) d\bar{z}}{\bar{z} - \bar{z}_0}.
\end{align*}
\]
(4.16) \[ F_\Pi (\zeta) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi (\zeta) - \varphi (\zeta_0)}{\zeta - \zeta_0} \, d\zeta + \frac{1}{2\pi i} \int_{\Gamma'_{\Pi \zeta_0}} \varphi (\zeta_0) \, d\zeta \]

Subtracting (4.15) from (4.16), we have

(4.17) \[ F_\Pi (\zeta) - F_\Pi (\zeta_0) = \frac{1}{2\pi i} \int_{\Gamma_{\Pi \zeta_0}} \frac{\varphi (\zeta) \, d\zeta}{\zeta - \zeta_0} = \varphi (\zeta_0). \]

We have not assumed the regularity of \( \varphi (\zeta) \), but we get the same result as (4.8).

Rewrite (4.15) as follows:

(4.18) \[ F_\Pi (\zeta) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi (\zeta) - \varphi (\zeta_0)}{\zeta - \zeta_0} \, d\zeta + \frac{1}{2\pi i} \int_{\Gamma'_{\zeta_0}} \varphi (\zeta_0) \, d\zeta \]

\[ + \frac{1}{2\pi i} \int_{\Gamma_{\zeta_0} \zeta} \varphi (\zeta) \, d\zeta + \frac{1}{2\pi i} \int_{\Gamma_{\zeta_0} \zeta} \varphi (\zeta_0) \, d\zeta . \]

\[ = \frac{1}{2\pi i} \int_{\Gamma} \varphi (\zeta) \, d\zeta + \frac{1}{2\pi i} \int_{\Gamma'_{\zeta_0}} \varphi (\zeta_0) \, d\zeta + \frac{1}{2\pi i} \int_{\Gamma_{\zeta_0} \zeta} \varphi (\zeta_0) \, d\zeta . \]

When \( \zeta \to 0 \)

\[ \int_{\Gamma_{\zeta_0} \zeta} \varphi (\zeta_0) \, d\zeta \]

converges, and approaches zero as a limit. This has been proved as (4.11). The third integral can be expressed as the following way:

\[ \frac{1}{2\pi i} \int_{\Gamma_{\zeta_0} \zeta} \varphi (\zeta_0) \, d\zeta = \frac{\varphi (\zeta_0) \Delta \ln (\zeta - \zeta_0)}{2\pi i} \frac{1}{\text{AbB}} . \]

Since \( \ln (\zeta - \zeta_0) = \ln \zeta + i \text{Arg} (\zeta - \zeta_0) \),

\[ \varphi (\zeta_0) \Delta \ln (\zeta - \zeta_0) = \frac{\varphi (\zeta_0) \Delta \ln \zeta}{2\pi i} \cdot \text{Arg} (\zeta - \zeta_0) = \frac{\varphi (\zeta_0)}{2\pi i} \frac{1}{\text{AbB}} . \]

\( 1(\text{AbB}) \) means the length of arc \( \text{AbB} \).

However, the arc \( \Gamma'_{\zeta_0} \) with \( A, B \) as end-points is contained within the oppositely vertical angles with \( \zeta_0 \) as vertex and its
bisector coincident with the tangent at \( z_0 \). As \( \rho \to 0 \), the opposite vertical angles become smaller and smaller; hence \( A \) and \( B \) separately approach the points of intersection of the tangent at \( z_0 \) and circumference of the circle \( |z - z_0| = \rho \).

Accordingly,

\[
\lim_{\rho \to 0} \frac{1}{2\pi i} \int_{\text{AbB}} \frac{\varphi(z) dz}{z - z_0} = \frac{\varphi(z_0)}{2\pi} \cdot \pi = \frac{\varphi(z_0)}{2}.
\]

Since the expression on the left side of (4.18) is independent of \( \rho \), we can draw the conclusion that

\[
(4.19) \quad \lim_{\rho \to 0} \frac{1}{2\pi i} \int_{r} \frac{\varphi(z) dz}{z - z_0}
\]

exists and we denote it by

\[
(4.20) \quad F(I) = \frac{1}{2\pi i} \int_{r} \frac{\varphi(z) dz}{z - z_0} + \frac{1}{2} \varphi(z_0)
\]

Combining with (4.17), we have again

\[
(4.21) \quad F(\bar{z}) = \frac{1}{2\pi i} \int_{r} \frac{\varphi(z) dz}{\bar{z} - \bar{z}_0} - \frac{1}{2} \varphi(z_0).
\]
BIBLIOGRAPHY


2. Маркушевич, А. И., Теория аналитических функций, Russia, 1950.