Optimum problem involving integral equations

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OPTIMUM PROBLEM INVOLVING INTEGRAL EQUATIONS

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Extremum problem involving certain linear integral equations with linear constraints is considered. We first give a discussion of integral equation with emphasis on the resolvent. Then, we use the resolvent method to convert an extremum problem involving a state and control into one where only the control appears. In the form involving just the controls, the optimal controls are easy to characterize.
PREFACE

This thesis is aimed at providing a clear and brief introduction to integral equations and then investigating extremum problems governed by linear integral equations.

The thesis is divided into four chapters. The first chapter introduces the concept of integral equations. It will introduce the reader to the Fredholm and Volterra equations.

The second chapter deals with solutions of integral equations. It introduces the Laplace transforms method, the successive approximations method and the resolvent kernel method. However, only the resolvent kernel method will be explained in detail since we will use it later on.

The third and fourth chapters deal with integral operators and optimization problems respectively. However, a special interest will be given to the optimization problems, main subject of our thesis.

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CHAPTER ONE

CONCEPTS FROM INTEGRAL EQUATIONS

1.1. Fredholm Equations

We will consider integral equations of the form

\[ x(t) - \int_a^b K(t,s)x(s)ds = f(t) \]  \hspace{1cm} (1.1.1)

where the function \( x(t) \) is unknown, and the functions \( f(t) \)
and \( K(t,s) \) are assumed to be given. Moreover, \( a \) and \( b \) are
in general constants and can be either finite or infinite. We also suppose that the variable \( t \) varies in the interval
\((a,b)\), which is referred to as a basic interval. The function \( f(t) \) is called the absolute term of the integral equa-
tion (1.1.1) and the function \( K(t,s) \) its kernel. The kernel
\( K(t,s) \) is defined in the \((t,s)\)-plane in the square \( a < t, s < b \), sometimes called a basic square.

In our discussion we will be considering not one inte-
gral equation but the family of these equations

\[ x(t) - \lambda \int_a^b K(t,s)x(s)ds = f(t) \]  \hspace{1cm} (1.1.2)

where \( \lambda \) is an arbitrary numerical quantity, called parameter
of the equation (1.1.2). Note also that the variables \( t \) and
\( s \) are regarded real, even if the parameter \( \lambda \) and the func-
tions \( x(t) \), \( f(t) \) and \( K(t,s) \) can take real as well as complex
values.

Equation of the type (1.1.2) belongs to a class of
linear equations. It is said to be homogeneous if \( f(t) \equiv 0 \)
and nonhomogeneous if \( f(t) \neq 0 \).

It is assumed most of the time that all integrals considered here are to be taken in the Lebesgue sense and all functions under consideration are measurable in the sense of Lebesgue.

The equation (1.1.2) is called a Fredholm equation of the second kind if its kernel and absolute term are quadratically summable (integrable), the former in the basic square and the latter in the basic interval. An equation of the form

\[
\int_a^b K(t,s)x(s)\,ds = f(t)
\]

with the same assumptions on the kernel and the absolute term is called Fredholm equation of the first kind.

According to the definition of the Fredholm equation, its kernel is subjected to the condition

\[
\int_a^b \int_a^b |K(t,s)|\,dtds = B^2 < \infty \quad (1.1.3)
\]

Thus, it follows, using the Fubini theorem, that

\[
\int_a^b |K(t,s)|^2\,ds
\]

exists for almost all \( x \in (a,b) \) and is summable in \( (a,b) \) and

\[
\int_a^b |K(t,s)|^2\,dt
\]

is defined for all \( t \in (a,b) \). In most cases, we impose the following additional condition on the kernel: There exists a constant \( A \) such that the inequality
\[ \int_a^b |K(t,s)|^2 ds \leq A \quad (1.1.4) \]

is satisfied for all \( x \in (a,b) \). If the interval \((a,b)\) is finite, then satisfaction of condition (1.1.4) implies that of condition (1.1.3); in this case the constants \( A \) and \( B \) are connected by the relation

\[ B^2 \leq A(b-a) \quad (1.1.5) \]

In the case of infinite interval, the conditions (1.1.4) and (1.1.3) are independent.

### 1.2. Volterra Equations

Consider now an integral equation of the form

\[ x(t) - \lambda \int_a^b K(t,s)x(s) ds = f(t) \quad (1.2.1) \]

where \( b(t) \) is a function of \( t \) and not a constant as in (1.1.1). Such integral equations are called Volterra integral equations of the second kind when \( b(t) = t \), i.e.,

\[ x(t) - \lambda \int_a^t K(t,s)x(s) ds = f(t) \quad (1.2.2) \]

When \( x(t) \equiv 0 \), it is called a Volterra integral equation of the first kind, that is

\[ -\lambda \int_a^t K(t,s)x(s) ds = f(t) \]

so that, the Volterra and Fredholm integral equations look very similar except for the limit \( b(t) = t \) in Volterra equation and \( b(t) = b \) (constant) in Fredholm equation.
1.3. Example

Integral equations appear in a number of problems related to different fields such as physics, differential equations, population dynamics, elasticity, etc. Now, to illustrate the prevalence of integral equations, let us look at an example given in [4], pp. 2-3: The problem of forecasting human population may be one of the clearest examples formulated as an integral equation since the population \( n(t) \) at time \( t \) depends on all children born during the time interval \( 0 < \tau < t \) who survive to time \( t \). This dependency of the population \( n(t) \) on previous populations \( n(\tau) \), \( 0 < \tau < t \), is given by the integral equation

\[
    n(t) = n_0 f(t) + K \int_0^t f(t-\tau)n(\tau)d\tau \tag{1.3.1}
\]

where \( n_0 \) is the number of children born at time \( t = 0 \) and \( f(t) \) is the survival function (Fig. 1.1), which is the fraction of the number of children born at \( t = 0 \) that survive to age \( t \). With regard to the integral in (1.3.1) we may remark that \( K f(t-\tau)n(\tau) \Delta \tau \) represents the number of children born in the time interval \( \Delta \tau \) around time \( \tau \) that survive to time \( t \). It is clear that their number is proportional to \( n(\tau) \), the population present at time \( \tau \), and that their survival function at time \( t \) is \( f(t-\tau) \) since they are then of age \( t-\tau \).
Fig. 1.1. The Survival Function (From Jerri, 1982; Courtesy of COMAP, Inc.).
CHAPTER TWO

WAYS OF SOLVING INTEGRAL EQUATIONS

In this chapter we will discuss some of the methods of solving integral equations namely the Laplace transform method, the successive approximations method and the Neumann series method. However, a special emphasis will be on Neumann series and from now on we will discuss only integral equations of the second kind.

2.1. Laplace Transforms Method

Let us consider the integral equation

$$x(t) = f(t) + \int_0^t K(t, \tau)x(\tau)\,d\tau$$  \hspace{1cm} (2.1.1)

The method of Laplace transforms is used to solve integral equations when $K(t, \tau)$ is a difference kernel, that is, $K(t, \tau) = K(t-\tau)$. Thus, (2.1.1) becomes

$$x(t) + f(t) + \lambda \int_0^t K(t-\tau)x(\tau)\,d\tau$$  \hspace{1cm} (2.1.2)

Now observing that

$$\int_0^t K(t-\tau)x(\tau)\,d\tau = K^*x,$$

and taking the Laplace transforms of both sides in (2.1.2), we have

$$L[x(t)] = L[f(t)] + \lambda L[K^*x]$$  \hspace{1cm} (2.1.3)

However,
\[ L[K^*x] = L[K] \cdot L[x] \]

Hence, (2.1.3) becomes

\[ L[x(t)] = L[f(t)] + \lambda L[K] \cdot L[x] \]

Therefore,

\[ L[x(t)] = \frac{L[f(t)]}{1 - \lambda L[K]} \]

So that, the solution \( x(t) \) of (2.1.1) is given by

\[ x(t) = L^{-1} \left[ \frac{L[f(t)]}{1 - \lambda L[K]} \right] \quad (2.1.4) \]

Example 1: Use the Laplace transform method to solve

\[ t \quad y(t) + \int_{0}^{t} y(\tau)(t-\tau) \, d\tau = t \quad (2.1.5) \]

Clearly, \( K(t,\tau) = t-\tau. \)

Taking the Laplace transform in (2.1.5), we have

\[ L[y] + L[\int_{0}^{t} y(\tau)(t-\tau) \, d\tau] = L[t] \]

Thus,

\[ L[y] + L[y(t) \cdot K(t)] = L[t] \]

Hence,

\[ L[y] + L[y] \cdot L[t] = L[t] \]

Therefore,

\[ L[y] = \frac{1}{s^2 + 1} \]
so that,
\[ y(t) = L^{-1} \left[ \frac{1}{s^2 + 1} \right] = \sin t \]

Example 2: Solve the integral equation
\[ y(t) = t - 4 \int_0^t y(\tau) d\tau - 13 \int_0^t (t-\tau)y(\tau) d\tau \]  
\[ (2.1.6) \]

Taking the Laplace transform on both sides of (2.1.6) and using the same technique as above, we get
\[ L[y] = L[t] - 4L[\int_0^t y(\tau) d\tau] - 13L[\int_0^t (t-\tau)y(\tau) d\tau] \]
\[ = \frac{1}{s^2} - 4L[\ast y(t)] - 13L[t \ast y(t)] \]
\[ = \frac{1}{s^2} - 4 \frac{1}{s} L[y] - 13 \cdot \frac{1}{s^2} L[y] \]

so that
\[ L[y] = \frac{1}{s^2 + 4s + 13} = \frac{1}{(s+2)^2 + 9} \]

and
\[ y(t) = L^{-1} \left[ \frac{1}{(s+2)^2 + 9} \right] = \frac{1}{3} L^{-1} \left[ \frac{3}{(s+2)^2 + 9} \right] = \frac{1}{3} e^{-2t} \sin 2t \]

2.2. Method of Successive Approximations

Consider the integral equation
\[ x(t) = f(t) + \lambda \int_a^t K(t,\tau)x(\tau) d\tau \]  
\[ (2.2.1) \]

and assume that the function \( f(t) \) is continuous for \( 0 \leq t \leq a \) and the function \( K(t,\tau) \) is continuous for \( 0 \leq t \leq a \) and \( 0 \leq \tau \leq t \).
Let \( x_0(t) \) be an initial approximation for the integral in (2.2.1). Then, the first approximation \( x_1(t) \) is given by

\[
x_1(t) = f(t) + \int_0^t K(t,\tau)x_0(\tau)d\tau
\]

For the second, third, ... , \( n^{th} \) approximations, we proceed as follows:

\[
x_2(t) = f(t) + \lambda^2 \int_0^t K(t,\tau)x_1(\tau)d\tau
\]

\[
x_3(t) = f(t) + \lambda^3 \int_0^t K(t,\tau)x_2(\tau)d\tau
\]

\[...
\]

\[
x_n(t) = f(t) + \lambda^n \int_0^t K(t,\tau)x_{n-1}(\tau)d\tau
\]

Hence, the sequence \( \{x_n(t)\} \) obtained from successive approximations of (2.2.1), will converge to the solution \( x(t) \) of (2.2.1).

Now, let us apply the successive approximations method to solve the equation (2.1.5). Notice first that \( f(t) = t \) and \( K(t,\tau) = t-\tau \) and that both are continuous everywhere. If we start with \( y_0(t) = 0 \), we have

\[
y_1(t) = t - 0 = t
\]

\[
y_2(t) = t - \int_0^t (t-\tau)y_1(\tau)d\tau = t - \int_0^t (t-\tau)d\tau = t - \frac{t^3}{3!}
\]

\[
y_3(t) = t - \int_0^t (t-\tau)y_2(\tau)d\tau = t - \int_0^t (t-\tau)(\tau - \frac{t^3}{3!})d\tau = t - \frac{t^3}{3!} + \frac{t^5}{5!}
\]
Thus,
\[ y_n(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots + (-1)^n \frac{t^{2n+1}}{(2n+1)!} \]

Therefore, the sequence \( \{y_n(t)\} \) converges to \( \sin t \), which is exactly the solution found in Example 1 of Section 2.1.

2.3. Neumann Series Method

This method is also known as the resolvent kernel method. Let us consider a Fredholm integral equation of the second kind

\[ x(t) = \lambda \int_a^b K(t,s)x(s)ds + b(t) \]  

(2.3.1)

As usual, the kernel \( K(t,s) \) is assumed to be an integrable function of \( t \) and \( s \) \( (a \leq t, s \leq b) \) and the solution of equation (2.3.1) is sought in the form of a series

\[ x(t) = x_0(t) + \lambda x_1(t) + \lambda^2 x_2(t) + \ldots + \lambda^n x_n(t) + \ldots \]  

(2.3.2)

If the series converges uniformly for some values of the parameter \( \lambda \), it may be substituted in (2.3.1); and by equating the coefficients of like powers of \( \lambda \), we obtain the recurrence relations

\[ x_i(t) = \int_a^b K(t,s)x_{i-1}(s)ds \quad (i = 1,2,3,\ldots) \]  

(2.3.3)

\[ x_0(t) = b(t) \]

Assume that \( K(t,s) \) and \( b(t) \) are bounded, that is, \( |K(t,s)| < A, \ |b(t)| < M \) where \( A,M \) are constants. It follows that
Thus, the series \(2.3.2\) converges if

\[
|\lambda| < \frac{1}{A(b-a)}
\]  

(2.3.4)

Moreover, if we use \(2.3.3\), we get for the series \(x(t)\),

\[
x(t) = b(t) + \lambda \left\{ \int_a^b K_1(t,s)b(s)ds + \lambda \int_a^b K_2(t,s)b(s)ds + \lambda^2 \int_a^b K_3(t,s)b(s)ds + \ldots \right\}
\]  

(2.3.5)

where \(K_n(t,s)\), called iterated kernels, are related by the equations

\[
K_n(t,s) = \int_a^b K(t,u)K_{n-1}(u,s)du, \quad (n = 2,3,\ldots)
\]  

(2.3.6)

\[
K_1(t,s) = K(t,s)
\]

The same condition \(2.3.4\) implies that the following series

\[
K_1(t,s) + \lambda K_2(t,s) + \lambda^2 K_3(t,s) + \ldots = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(t,s) = \Gamma(t,s;\lambda)
\]  

(2.3.7)

converges and \(\Gamma(t,s;\lambda)\) is known as the resolvent of equation \((2.3.1)\). Thus, \(2.3.5\) becomes, using the representation in \(2.3.7\),

\[
x(t) = b(t) + \lambda \int_a^b \Gamma(t,s;\lambda)b(s)ds
\]

\[
= \sum_{i=1}^{\infty} \lambda^i \int_a^b K_i(t,s)b(s)ds + b(t)
\]  

(2.3.8)

And \(2.3.8\) is referred to as the Neumann series which
represents in fact the solution of the integral equation given by (2.3.1) since the sequence \( \{x^n(t)\} \), where

\[
x_1(t) = \lambda \int_a^b K(t,s)b(s)ds + b(t)
\]

\[
x_2(t) = \lambda^2 \int_a^b \int_a^b K(t,s)K(s,u)b(u)duds
\]

\[
+ \lambda \int_a^b K(t,s)b(s)ds + b(t)
\]

\[
= \lambda^2 \int_a^b K^2(t,s)b(s)ds + \lambda \int_a^b K(t,s)b(s)ds + b(t)
\]

\[
: : : : \]

\[
x_n(t) = \sum_{i=1}^{n} \lambda^i \int_a^b K_i(t,s)b(s)ds + b(t),
\]

converges uniformly to \( x(t) = \sum_{i=1}^{\infty} \lambda^i \int_a^b K_i(t,s)b(s)ds + b(t) \) by the Banach fixed point theorem. Thus, knowing the resolvent, we at once obtain the solution of the original equation (2.3.1) for any \( \lambda \) satisfying (2.3.4).

Now, observe that if all \( K_n(t,s) \) are expressed in terms of \( K(t,s) \) of the original equation, (2.3.6) becomes

\[
K_n(t,s) = \int_a^b \int_a^b \ldots \int_a^b K(t,u_{n-1})K(u_{n-1},u_{n-2}) \ldots K(u_1,s)du_1 \cdot du_2 \ldots du_{n-1}
\]

(2.3.9)

\[
K_{p+q}(t,s) = \int_a^b K_p(t,u)K_q(u,s)du
\]

(2.3.10)

And in the special case of \( p = n-1, q = 1; \) i.e., \( p+q = n \), we obtain
Kn(t,s) = \int_a^b K_{n-1}(t,u)K(u,s)du \quad (n = 2,3,\ldots) \quad (2.3.11)

K_{n+1}(t,s) = \int_a^b K_n(t,u)K(u,s)du \quad (n = 1,2,\ldots)

Now, in the representation (2.3.7) of the resolvent \( \Gamma(t,s;\lambda) \), assuming \( \lambda \) to be sufficiently small, let us consider a chain of equalities

\[
\Gamma(t,s;\lambda) = K_1(t,s) + \lambda K_2(t,s) + \lambda^2 K_3(t,s) + \ldots
\]

\[
= K(t,s) + \lambda \int_a^b K(t,u)K_1(u,s)du
\]

\[
+ \lambda^2 \int_a^b K(t,u)K_2(u,s)du + \ldots
\]

\[
= K(t,s) + \lambda \int_a^b K(t,u[K_1(u,s) + \lambda K_2(u,s) + \ldots]du
\]

\[
= K(t,s) + \lambda \int_a^b K(t,u)\Gamma(u,s;\lambda)du \quad (2.3.12)
\]

This relation may be regarded as a functional equation for the resolvent. However, using formula (2.3.11), we get a different functional equation for the resolvent, that is

\[
\Gamma(t,s;\lambda) = K(t,s) + \lambda \int_a^b K(u,s)[K_1(t,u) + \lambda K_2(t,u) + \ldots]du
\]

\[
= K(t,s) + \lambda \int_a^b K(u,s)\Gamma(t,u;\lambda)du \quad (2.3.13)
\]

Note, however, that in the above discussion, the resolvent has been defined only upon fulfillment of condition (2.3.4), that is, the resolvent is an analytic function of the parameter \( \lambda \) in the circle \(|\lambda| < \frac{1}{A(b-a)}\). Equations (2.3.12) and (2.3.13) allow the resolvent to be determined in the whole plane of the complex variable, with the exception
of some values. Suppose now that a function $\Gamma(t,s;\lambda)$ exists in the square $a \leq t, s \leq b$, prescribed for a certain value of $\lambda$ and satisfying relations (2.3.12) and (2.3.13). Let us show that (2.3.1) has a solution representable in the form of (2.3.8). Observe that multiplying both sides of (2.3.1) by $\lambda \Gamma(s,t;\lambda)$ and integrating with respect to $t$ and finally, transforming the resulting expression with the use of (2.3.13), we get

$$
\frac{b}{a} \int b(t)\Gamma(s,t;\lambda)dt - \frac{b}{a} \int K(t,s)x(s)ds = 0,
$$

which, with (2.3.1), leads to the required representation

$$
x(s) = \lambda \Gamma(s,t;\lambda)b(t)dt + b(s)
$$

or

$$
x(t) = \lambda \Gamma(t,s;\lambda)b(s)ds + b(t)
$$

It remains to show that a function representable by expression (2.3.8) is a solution of (2.3.1). Indeed, by substituting (2.3.8) in (2.3.1), we arrive at an identity.

**Example 1:** Given the integral equation

$$
x(t) = \frac{1}{e^{t-s}} x(s)ds + b(t)
$$

we want to give its solution using the resolvent method.

Comparing with (2.3.1), we see that $a = 0$, $b = 1$, $K(t,s) = e^{t-s}$. Thus
\[ K_1(t,s) = K(t,s) = e^{t-s} \]

\[ K_2(t,s) = \int_0^1 K(t,u)K_1(u,s)\,du \]

\[ = \int_0^1 e^{t-s} u^s du = e^{t-s} = K(t,s) \]

\[ K_3(t,s) = \int_0^1 K(t,u)K_2(u,s)\,du \]

\[ = \int_0^1 e^{t-s} du = e^{t-s} = K(t,s) \]

\[ \vdots \]

\[ K_n(t,s) = e^{t-s} = K(t,s) \]

Hence

\[ \Gamma(t,s;\lambda) = K_1(t,s) + \lambda K_2(t,s) + \ldots \]

\[ = K(t,s) [1 + \lambda + \lambda^2 + \ldots] \]

\[ = K(t,s) \sum_{n=0}^{\infty} \lambda^n \]

\[ = \frac{e^{t-s}}{1-\lambda}, \quad |\lambda| < 1 \]

so that, our integral equation has one and only one solution

for \(|\lambda| < 1\) which is given by

\[ x(t) = b(t) + \lambda \int_0^1 \Gamma(t,s,\lambda) b(s)\,ds \]

\[ = b(t) + \frac{\lambda}{1-\lambda} \int_0^1 e^{t-s} b(s)\,ds \]

\[ = b(t) + \frac{\lambda e^t}{1-\lambda} \int_0^1 e^{-s} b(s)\,ds \]

**Example 2:** Consider the Volterra integral equation of the second kind
\[ u(x) = g(x) + \lambda^2 \int_0^x (x-t)u(t)dt \]

Clearly \(K(x,t) = x-t\). Thus,

\[ K_1(x,t) = x - t \]

\[ K_2(x,t) = \int_t^x K(x,u)K_1(u,t)du \]
\[ = \int_t^x (x-u)du \]
\[ = \frac{(x-t)^3}{3!} \]

\[ K_3(x,t) = \int_t^x K(x,u)K_2(u,t)du \]
\[ = \int_t^x (x-u)\cdot\frac{(u-t)^3}{3!}du \]
\[ = -\frac{(x-t)^5}{5!} \]

\[ \vdots \]

\[ K_n(x,t) = (-1)^n \frac{(x-t)^{2n-1}}{(2n-1)!} \quad (n = 1,2,3,...) \]

Hence

\[ r(x,t;\lambda) = \sinh \lambda(x-t) = \frac{e^{\lambda(x-t)} - e^{-\lambda(x-t)}}{2}, \]

so that,

\[ u(x) = g(x) + \lambda \int_0^x \sinh \lambda(x-t)g(t)dt. \]
CHAPTER THREE

INTEGRAL OPERATOR

Now we will be interested in the concept of integral operator and its application to Fredholm equation.

3.1. Concept of Integral Operator

In the following, we will be concerned with the operator $I - \lambda K$, where $I$ is the identical operator and $K$ is any operator. Obviously, $I - \lambda K$ can be regarded as a polynomial of first degree with respect to $K$. If its inverse exists, $\lambda$ is called a regular value of $K$. For $\lambda \neq 0$, we then define the resolvent of $K$ at the point $\lambda$, denoted by $R_\lambda(K)$ or $\Gamma(t,s;\lambda)$ in integral equations, by

$$R_\lambda = -\lambda^{-1}(I - (I-\lambda K)^{-1}),$$

so that,

$$(I-\lambda K)^{-1} = I + \lambda R_\lambda$$  \hspace{1cm} (3.1.1)

If we set $R_0 = K$, (3.1.1) also holds for $\lambda = 0$. Since

$$(I-\lambda K)^{-1}(I-\lambda K) = (I-\lambda K)(I-\lambda K)^{-1} = I,$$

it follows from (3.1.1) that

$$R_\lambda - K = \lambda KR_\lambda = \lambda R_\lambda K$$  \hspace{1cm} (3.1.2)

Conversely, if there exists an operator $R_\lambda$ satisfying (3.1.2), then $\lambda$ is clearly a regular value and $(I-\lambda K)$ has the inverse
$I + \lambda R_\lambda$.

3.2. Application to Fredholm Equation

**Equations satisfied by the resolvent.**

Let $z = I - \lambda K$ be an integral operator such that

$$z(s) = W(s) - \lambda \int_0^1 K(s,t)W(t)dt$$

We show that the resolvent kernel $\Gamma(s,t;\lambda)$ with

$$(I-\lambda K)^{-1}W = W + \lambda \int_0^1 \Gamma(s,t;\lambda)W(t)dt$$

must satisfy

$$\Gamma(s,t;\lambda) = K(s,t) + \lambda \int_0^1 K(s,u)\Gamma(u,t;\lambda)du$$

For that, let $\Gamma(s,t;\lambda)$ be the resolvent kernel of the integral equation

$$W(s) = \lambda \int_0^1 K(s,t)W(t)dt + z(s) \quad (3.2.1)$$

Clearly,

$$W = (I-\lambda K)^{-1}z = (I+\lambda R_\lambda)z$$

where $R_\lambda = \Gamma(s,t;\lambda)$. Thus, the solution of (3.2.1) is given by, using (2.3.8),

$$W(s) = z(s) + \lambda \int_0^1 \Gamma(s,t;\lambda)z(t)dt$$

$$= \sum_{i=1}^{\infty} \lambda^i \int_0^1 K_i(s,t)z(t)dt,$$
where $K_1(s,t)$ are the iterated kernels. But considering that the resolvent

$$
\Gamma(s,t;\lambda) = K_1(s,t) + \lambda K_2(s,t) + \lambda^2 K_3(s,t) + \ldots
$$

and taking into consideration that

$$
\frac{1}{\Gamma(s,t;\lambda)} = K(s,t) + \lambda \int K(s,u)K_1(u,t)du + \ldots + \lambda^n \int K(s,u)K_n(u,t)du + \ldots
$$

Hence

$$
\frac{1}{\Gamma(s,t;\lambda)} = K(s,t) + \lambda \int K(s,u)\Gamma(u,t;\lambda)du.
$$

(3.2.3)

Now observe that (3.2.2) can also be written as

$$
K_n(s,t) = \int K_{n-1}(s,u)K(u,t)du, \quad n = 2, 3, \ldots
$$

Thus,
\[ \Gamma(s,t;\lambda) = K(s,t) + \lambda \int_0^1 K_1(s,u)K(u,t)du \\
+ \lambda^2 \int_0^1 K_2(s,u)K(u,t)du + \ldots \\
+ \lambda^n \int_0^1 K_n(s,u)K(u,t)du + \ldots \\
= K(s,t) + \lambda \int_0^1 [K_1(s,u) + \lambda K_2(s,u) + \ldots \\
+ \lambda^{n-1} K_n(s,u) + \ldots ]K(u,t)du \\
= K(s,t) + \lambda \int_0^1 \Gamma(s,u;\lambda)K(u,t)du \quad (3.2.4) \]

so that, from (3.2.3) and (3.2.4), we have

\[ \Gamma(s,t;\lambda) = K(s,t) + \lambda \int_0^1 K(s,u)\Gamma(u,t;\lambda)du \\
= K(s,t) + \lambda \int_0^1 \Gamma(s,u;\lambda)K(u,t)du. \]
CHAPTER FOUR

OPTIMIZATION PROBLEM

4.1. Introduction: The Problem

In this chapter, we will be interested in a control problem governed by an integral equation [6]. Let us suppose we have a control system governed by

$$\phi(t) = f(t) + \int_0^t L(t,\phi(s),u(s),s)\,ds.$$  \hspace{1cm} (4.1.1)

We would like to find a control $u(\cdot)$ such that (4.1.1) and

$$\int_0^1 M(\phi(t),u(t),t)\,dt + W(\phi(1)) = 0$$  \hspace{1cm} (4.1.2)

are satisfied and the cost function

$$\int_0^1 f^0(\phi(t),u(t),t)\,dt$$  \hspace{1cm} (4.1.3)

is minimum where $u(t) \in K$, $K$ being a convex polyhedron in $\mathbb{R}^m$. However, we will be concerned with a problem where the optimal relaxed controls turn out to be ordinary controls:

$$\min \int_0^1 [a(s) \cdot \phi(s) + b(s) \cdot u(s)]\,ds$$  \hspace{1cm} (4.1.4)

subject to

$$\phi(t) = f(t) + \int_0^t A(t,s)\phi(s)\,ds + \int_0^t B(t,s)u(s)\,ds, (4.1.5)$$

$$\int_0^1 C(t) \cdot \phi(t)\,dt + \int_0^1 D(t) \cdot u(t)\,dt + W(\phi(1)) = 0 \hspace{1cm} (4.1.6)$$

where $A(t,s)$ is an $n \times n$ matrix and $B(t,s)$ is an $n \times m$ matrix. These matrices are continuous in both variables and
continuously differentiable in $t$. Finally $a(s)$, $b(s)$, $C(t)$ and $D(t)$ are continuous vectors of appropriate dimension. The optimal pairs $(\phi, u)$ are shown to satisfy the following conditions [6]: If $(\phi_0, u_0)$ is optimal for (4.1.4)-(4.1.6), then there exists a square integrable function $z$, scalars $\lambda^0 \geq 0$ and $\xi$ such that

(i) \[ ||z||_{\infty} + \lambda^0 + |\xi| \neq 0 \] (4.1.7)

(ii) \[ \int_0^1 z(s)A(s,t)ds \] (4.1.8)

(iii) \[ \int_0^1 z(s)B(t,s)u(s)ds \]

(iv) \[ u_0 \text{ takes values in the vertices of } K. \] (4.1.10)

4.2. Linear Case of the Optimization Problem

Let us consider the problem

Min \[ \int_0^1 [a(s)x(s) + b(s)u(s)]ds \] (4.2.1)

Subject to

\[ x(t) = f(t) + \int_0^t A(t,s)x(s)ds + \int_0^t B(t,s)u(s)ds, \] (4.2.2)

\[ \int_0^1 [C(t)x(t)dt + D(t)u(t)]dt + W(x(1)) = 0. \] (4.2.3)
We will confirm in the following discussion that $u_0$ takes values in the vertices of $K$, a convex polyhedron in $\mathbb{R}^m$, as predicted in (4.1.10) of Section 4.1. For simplicity, we consider one dimensional problem and take the control set $K$ to be the interval $[a, b]$.

Now observe that (4.2.2) is a Volterra equation of the second kind and may be written as

$$x(t) = f(t) + \int_0^t B(t,s)u(s)ds + \int_0^t A(t,s)x(s)ds, \quad (4.2.4)$$

and using the Neumann series technique, we get

$$x(t) = f(t) + \int_0^t B(t,s)u(s)ds$$

$$+ \int_0^t R(t,s)\left[\int_0^s B(s,\tau)u(\tau)d\tau + f(s)\right]ds$$

where $R(t,s) = R(t,s;1)$ is the resolvent kernel of (4.2.4).

Thus,

$$x(t) = f(t) + \int_0^t B(t,s)u(s)ds + \int_0^t \int_0^s R(t,s)B(s,\tau)u(\tau)d\tau ds$$

$$+ \int_0^t R(t,s)f(s)ds$$

$$= f(t) + \int_0^t B(t,s)u(s)ds + \int_0^t \int_0^\tau R(t,s)B(s,\tau)u(\tau)d\tau ds$$

$$+ \int_0^t R(t,s)f(s)ds$$

$$= f(t) + \int_0^t R(t,s)f(s)ds + \int_0^t B(t,s)u(s)ds$$

$$+ \int_0^t \int_0^\tau R(t,s)B(s,\tau)u(\tau)d\tau ds$$

Hence, letting
\[ \tilde{x}(t) = f(t) + \int_0^t R(t,s)f(s)\,ds \]

we have

\[ x(t) = \tilde{x}(t) + \int_0^t B(t,\tau)u(\tau)\,d\tau \]
\[ + \int_0^t [\int_0^\tau R(t,s)B(s,\tau)\,ds]u(\tau)\,d\tau \]
\[ = \tilde{x}(t) + \int_0^t [B(t,\tau) + \int_0^\tau R(t,s)B(s,\tau)\,ds]u(\tau)\,d\tau \]
\[ = \tilde{x}(t) + \int_0^t R_1(t,\tau)u(\tau)\,d\tau \quad (4.2.5) \]

where

\[ R_1(t,\tau) = B(t,\tau) + \int_\tau^t R(t,s)B(s,\tau)\,ds \]

Now let us analyze the second condition of our minimization problem, namely (4.2.3). Using (4.2.5) we have

\[ \frac{1}{t} \int_0^t C(t)\tilde{x}(t)\,dt + \int_0^t \int_0^\tau C(t)R_1(t,\tau)u(\tau)\,d\tau\,dt \]
\[ + \int_0^t D(t)u(t)\,dt + W(x(l)) = 0, \]
changing the order of integration we obtain

\[ \frac{1}{t} \int_0^t C(t)\tilde{x}(t)\,dt + \int_0^t \int_0^\tau [C(t)R_1(t,\tau)dt + D(\tau)]u(\tau)\,d\tau \]
\[ + W(x(l)) = 0, \quad (4.2.6) \]

and using (4.2.5) and letting \( t = l, \ W(x(l)) = W \cdot x(l) \), we have
\[ W(x(1)) = W \cdot x(1) \]
\[ = W \cdot \left[ \tilde{f}(1) + \int_0^1 R_1(t, \tau) u(\tau) d\tau \right] \]
\[ = W \cdot f(1) + \int_0^1 W \cdot R(1, \tau) f(\tau) d\tau \]
\[ + \int_0^1 W \cdot R_1(1, \tau) u(\tau) d\tau \]

Hence, (4.2.6) becomes
\[ \int_0^1 C(t) \tilde{f}(t) dt + \int_0^1 \int_0^1 C(t) R_1(t, \tau) dt + D(\tau) u(\tau) d\tau \]
\[ + W \cdot f(1) + \int_0^1 W \cdot R(1, \tau) f(\tau) d\tau \]
\[ + \int_0^1 W \cdot R_1(1, \tau) u(\tau) d\tau = 0 \]

or
\[ \int_0^1 \int_0^1 C(t) R_1(t, \tau) dt + D(\tau) + W \cdot R_1(1, \tau) u(\tau) d\tau \]
\[ + \int_0^1 C(t) \tilde{f}(t) dt + W \cdot f(1) \]
\[ + \int_0^1 W \cdot R(1, \tau) f(\tau) d\tau = 0. \]

Setting
\[ \int_0^1 C(t) R_1(t, \tau) dt + D(\tau) + W \cdot R_1(1, \tau) = R_2(\tau), \quad (4.2.7) \]
and
\[ \int_0^1 C(t) \tilde{f}(t) dt + W \cdot f(1) + \int_0^1 W \cdot R(1, \tau) f(\tau) d\tau = C_1, \]
we have
\[ \int_0^1 R_2(\tau) u(\tau) d\tau + C_1 = 0 \]
or

\[
\frac{1}{\rho} \int_{\tau}^{t} [R_2(\tau)u(\tau) + C_1]d\tau = 0. \quad (4.2.8)
\]

so that, using (4.2.5) in (4.2.1) we have

\[
\frac{1}{\rho} \int_{\tau}^{t} [a(s)x(s) + b(s)u(s)]ds
\]

\[
= \frac{1}{\rho} \int_{\tau}^{t} [a(t)x(t) + b(t)u(t)]dt
\]

\[
= \frac{1}{\rho} \int_{\tau}^{t} [a(t)f(t) + \int_{\tau}^{T} R_1(t,\tau)u(\tau)d\tau + b(t)u(t)]dt
\]

\[
= \frac{1}{\rho} \int_{\tau}^{t} [a(t)f(t)dt + \int_{\tau}^{T} R_1(t,\tau)u(\tau)d\tau + b(t)u(t)]dt
\]

\[
= \frac{1}{\rho} \int_{\tau}^{t} a(t)f(t)dt + \frac{1}{\rho} \int_{\tau}^{T} R_1(t,\tau)u(\tau)d\tau + b(T)u(T)dt.
\]

Therefore, our problem (4.2.1)-(4.2.3) is reduced to

\[
\operatorname{Min}\{ \int_{\tau}^{t} a(t)f(t)dt + \int_{\tau}^{T} R_1(t,\tau)dt + b(\tau)u(\tau)\} (4.2.9)
\]

subject to

\[
\frac{1}{\rho} \int_{\tau}^{T} [R_2(\tau)u(\tau) + C_1]d\tau = 0. \quad (4.2.10)
\]

\[
\alpha \leq u(t) \leq \beta \quad (4.2.11)
\]

where

\[
\hat{f}(t) = f(t) + \int_{0}^{t} R(t,s)f(s)ds,
\]
\[ R_1(t, \tau) = B(t, \tau) + \int_0^t R(t, s)B(s, \tau)ds, \]
\[ R_2(\tau) = \int_0^1 C(t)R_1(t, \tau)dt + D(\tau) + W\cdot R_1(1, \tau), \]
\[ C_1 = \int_0^1 C(t)f(t)dt + W\cdot f(1) + \int_0^1 W\cdot R(1, \tau)f(\tau)dt. \]

Our task is now to find \( u \) such that \( a \leq u(t) \leq \beta \) for \( 0 \leq t \leq 1 \), (4.2.10) is satisfied and the expression between braces in (4.2.9) is minimized.

Example 1: Consider the Volterra integral equation of the second kind

\[ u(x) = f(x) + \lambda \int_0^x e^{x-t} u(t)dt \]

Clearly, its kernel and resolvent are as follows:

\[ K(x, t) = e^{x-t}, \]
\[ \Gamma(x, t; \lambda) = K_1(x, t) + \lambda K_2(x, t) + \lambda^2 K_3(x, t) + \ldots \]
\[ + \lambda^n K_{n+1}(x, t) + \ldots \]
\[ = e^{x-t} + \lambda(x-t)e^{x-t} + \lambda^2 \frac{(x-t)^2}{2!} e^{x-t} + \ldots \]
\[ + \lambda^n \frac{(x-t)^n}{n!} e^{x-t} + \ldots \]
\[ = e^{x-t} \left[ 1 + \lambda(x-t) + \lambda^2 \frac{(x-t)^2}{2!} + \ldots \right] \]
\[ + \lambda^n \frac{(x-t)^n}{n!} + \ldots \]
\[ = e^{x-t} e^{\lambda(x-t)} = e^{(1+\lambda)(x-t)} \]

Now, let us look at a particular case of (4.2.9) where
\[ B(t,s) = B, \quad C(t) = C, \quad D(t) = D, \quad A(t,s) = A; \quad A, B, C, D \text{ being constants and } K(t,s) = e^{t-s}. \] Then

\[
R_1(t, \tau) = B(t, \tau) + \int_{\tau}^{t} R(t, s)B(s, \tau)\,ds
= B + \int_{\tau}^{t} e^{2(t-s)}B\,ds
= \frac{B}{2} (1 + e^{2(t-\tau)})
\]

\[
R_2(\tau) = \int_{\tau}^{t} C(t)R_1(t, \tau)\,dt + D(\tau) + W \cdot R_1(1, \tau)
= \int_{\tau}^{t} C \frac{B}{2} (1 + e^{2(t-\tau)})\,dt + D + W \left[ \frac{B}{2} (1 + e^{2(1-\tau)}) \right]
= \frac{BC}{4} + \frac{WB}{2} e^{2(1-\tau)} + \frac{BC}{2} (1-\tau) - \frac{BC}{4} + D + \frac{WB}{2},
\]

and

\[
\tilde{f}(t) = f(t) + \int_{0}^{t} R(t, s)f(s)\,ds
= f(t) + \int_{0}^{t} e^{2(t-s)}f(s)\,ds
= f(t) + e^{2t} \int_{0}^{t} e^{-2s}f(s)\,ds.
\]

Letting \( f(t) = e^t \), we have

\[
\tilde{\tilde{f}}(t) = e^t + e^{2t} \int_{0}^{t} e^{-2s}e^s\,ds = e^{2t}.
\]

Thus, using the above results in the system (4.2.9)-(4.2.11), we have
\[ \int_0^1 a(t) \tilde{f}(t) \, dt + \int_0^1 \int_0^\tau a(t) R(t, \tau) \, dt + b(\tau) u(\tau) \, d\tau \]

\[ = \int_0^1 A e^{2t} \, dt + \int_0^\tau A B (1 + e^{2(t-\tau)} \, dt + B) u(\tau) \, d\tau \]

\[ = \frac{A}{2} (e^2 - 1) + \frac{B}{2} \int_0^\tau e^{2(1-\tau)} - A \tau + \frac{A}{2} + 2) u(\tau) \, d\tau, \]

and the equation (4.2.10) becomes

\[ \int_0^1 \left( \left[ \left( BC + WB \right) e^{2(1-\tau)} + \frac{BC}{2} (1-\tau) - \frac{BC}{4} + D + \frac{WB}{2} \right] u(\tau) + C_1 \right) \, d\tau = 0 \]

where

\[ C_1 = \int_0^1 C e^{2\tau} \, dt + W \cdot f(1) + \int_0^1 W \cdot R(1, \tau) f(\tau) \, d\tau \]

\[ = \frac{C}{2} (e^2 - 1) + W \cdot e + \int_0^1 \frac{B}{2} W \cdot (e^\tau + e^{2-\tau}) \, d\tau \]

Hence, the expression in braces in (4.2.9) becomes

\[ \int_0^1 \left( \left[ \left( BC + WB \right) e^{2(1-\tau)} + \frac{BC}{2} (1-\tau) - \frac{BC}{4} + D + \frac{WB}{2} \right] u(\tau) + \frac{B}{2} W \cdot (e^\tau + e^{2-\tau}) \, d\tau + \frac{C}{2} (e^2 - 1) + W \cdot e \right) \, d\tau = 0 \]

(4.2.14)

Now, taking \( C = 0 \), \( W = 0 \), \( D = 1 \), \( B = A = 2 \) in (4.2.13) and (4.2.14) and assuming that \(-1 \leq u(\tau) \leq 1\), our problem is reduced to

\[ \text{Min} \left\{ \int_0^1 (e^{2(1-\tau)} - 2\tau) u(\tau) \, dt \right\} \]

subject to

\[ \int_0^1 u(\tau) \, d\tau = 0, \quad -1 \leq u(\tau) \leq 1 \]

(4.2.15)
But observe in (4.2.15) that

\[ e^{2(1-t)} - 2t = 0 \quad \text{iff} \quad e^{2(1-t)} = 2t. \]

Thus, by looking at the graph of the functions \( e^{2(1-t)} \) and \( 2t \) below (Fig. 4.1)

![Graph](image)

Fig. 4.1. The Functions \( e^{2(1-t)} \) and \( 2t \).

We see clearly that there exists \( t_0 \) such that \( e^{2(1-t_0)} = 2t_0 \) and that \( e^{2(1-t)} - 2t \) is positive for \( 0 \leq t \leq t_0 \) and negative for \( t_0 < t \leq 1 \). Hence, combining this fact with the observation that \( e^{2(1-t)} - 2t \) decreases on \([0,1]\), we conclude that the optimal control \( u_0 \) is such that
Example 2: Consider the Volterra integral equation of the second kind

\[ u(x) = f(x) + \lambda \int_0^x u(t)dt \]  \hspace{1cm} (4.2.16)  

Here, the kernel \( K(x,t) \equiv 1 \). Thus, the resolvent kernel for (4.2.16) is given by

\[
\Gamma(x,t;\lambda) = K_1(x,t) + \lambda K_2(x,t) + \lambda^2 K_3(x,t) + \ldots
\]

\[
+ \lambda^{n+1} K_{n+1}(x,t)
\]

\[
= 1 + \lambda (x-t) + \lambda^2 \frac{(x-t)^2}{2!} + \ldots
\]

\[
+ \lambda^n \frac{(x-t)^n}{n!} + \ldots
\]

\[
= \sum_{n=0}^{\infty} \frac{\lambda^n (x-t)^n}{n!} = e^{\lambda (x-t)}
\]

Hence,

\[ u(x) = f(x) + \lambda \int_0^x e^{\lambda (x-t)}f(t)dt. \]

Now, looking at the system (4.2.9)-(4.2.11) with \( R(t,s) = e^{\lambda (x,t)} \) and letting \( \lambda = 1 \), \( B(t,s) = B \), \( C(t) = C \), \( D(t) = D \), \( A(t,s) = A \), \( f(t) = e^t \) as in Example 1, we have

\[
\tilde{f}(t) = f(t) + \int_0^t R(t,s)f(s)ds
\]

\[
= e^t + \int_0^t e^{(t-s)} e^sds = e^t(t+1),
\]
and

\[ R_1(t, \tau) = B(t, \tau) + \int_{\tau}^{t} R(t, s)B(s, \tau)\,ds \]
\[ = B + \int_{\tau}^{t} e^{t-s}Bds = Be^{t-\tau}, \]
\[ R_2(t, \tau) = \int_{\tau}^{1} C(t)R_1(t, \tau)\,dt + D(\tau) + W \cdot R_1(l, \tau) \]
\[ = \int_{\tau}^{1} CB \cdot e^{t-\tau}dt + D + W \cdot Be^{1-\tau} \]
\[ = (CB + WB)e^{1-\tau} - CB + D, \]

so that

\[ \int_{0}^{1} a(t)\tilde{f}(t)\,dt + \int_{0}^{1} [\int_{0}^{1} a(t)R_1(t, \tau)\,dt + b(\tau)]u(\tau)\,d\tau \]
\[ = \int_{0}^{1} Ae^{t(t+1)}\,dt + \int_{0}^{1} [\int_{0}^{1} ABE^{t-\tau}dt + B]u(\tau)\,d\tau \]
\[ = Ae + B \int_{0}^{1} (Ae^{1-\tau} - A + 1)u(\tau)\,d\tau, \]

and

\[ \int_{0}^{1} [R_2(\tau)u(\tau) + \int_{0}^{1} C(t)\tilde{f}(t)\,dt + W \cdot f(l) \]
\[ + \int_{0}^{1} W \cdot R(l, \tau)f(\tau)\,d\tau]d\tau = 0 \]
\[ \equiv \int_{0}^{1} [((CB + WB)e^{1-\tau} - CB + D)u(\tau) + \int_{0}^{1} Ce^{t(t+1)}\,dt \]
\[ + W \cdot e + \int_{0}^{1} W \cdot Be^{1-\tau}e\,d\tau]d\tau = 0 \]
\[ \equiv \int_{0}^{1} [((CB + WB)e^{1-\tau} - CB + D)u(\tau)\,d\tau + (C+W+WB)e = 0. \]

Thus, the system (4.2.9)-(4.2.10) becomes, letting \( \alpha = -1 \)
and \( \beta = 1, \)
\[
\text{Min}\left\{ B \int_0^1 (Ae^{1-\tau} - A + 1)u(\tau)d\tau \right\}
\]

subject to

\[
\int_0^1 [(CB+WB)e^{1-\tau} - CB + D]u(\tau)d\tau + (C+W+WB)e = 0,
\]

\[-1 \leq u(\tau) \leq 1\]

Now, observe that choosing \( A = -1 \) and \( B, C, D, W \) in such a way that

\[C + W = 0,\]
\[D - CB = 1,\]
\[WB = 0,\]

The system (4.2.17) becomes

\[
\text{Min}\left\{ B \int_0^1 (e^{-1+\tau} + 2)u(\tau)d\tau \right\}
\]

or

\[
\text{Min}\left\{ \int_0^1 (e^{-1+\tau} + 2)u(\tau)d\tau \right\}
\]

subject to

\[
\int_0^1 u(\tau)d\tau = 0, \quad -1 \leq u(\tau) \leq 1.
\]

But, observe also that

\[-e^{1-\tau} + 2 = 0\]

for \( \tau = 1 - \ln 2 \). Thus, \(-e^{1-\tau} + 2 < 0 \) for \( \tau < 1 - \ln 2 \) and \(-e^{1-\tau} + 2 > 0 \) for \( \tau > 1 - \ln 2 \). Hence, combining this fact
with the observation $2 - e^{1-\tau}$ is increasing on $[0,1]$, (see Fig. 4.2), we conclude that the optimal control $u_0(t)$ is such that

$$u_0(t) = \begin{cases} 
1, & 0 \leq t \leq 1/2 \\
-1, & 1/2 < t \leq 1
\end{cases}$$

![Graph of $f(\tau)$ showing $2 - e^{1-\tau}$ and key points](image)

**Fig. 4.2. The Function $2 - e^{1-\tau}$.**
BIBLIOGRAPHY


