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Contraction mappings and fixed point theorems

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CONTRACTION MAPPINGS AND FIXED POINT THEOREMS

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DEPARTMENT OF MATHEMATICS

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INTRODUCTION

This thesis is concerned with the solving of equations of the form $f(x) = x$, when $f$ satisfies certain conditions and $x$ takes on values in a complete metric space. (Incidentally, suppose we are dealing with a Banach Space). Then considering a solution of an equation of the form $g(x) = 0$, the zero element is equivalent to considering a solution of an equation of the form $f(x) = x$, where $g(x) = x - f(x)$. In mapping terminology, we shall consider a mapping $f:(X,d)\rightarrow(X,d)$, where $(X,d)$ is a complete metric space. We shall also seek a fixed point $x_0 \in X$ for $f$. A fixed point for $f$ is a point $x_0$ such that $f(x_0) = x_0$.

A theorem asserting the existence of a fixed point will be called a constructive fixed point theorem if its proof provides a constructive method for obtaining a fixed point. For example the existence of a solution may be established as the limit of a convergent sequence defined by an iterative procedure. The field of numerical functional analysis is largely concerned with constructive fixed point theorems and with iterative and approximation schemes for the solutions of linear and nonlinear equations for which
existence theorems are proved.

This thesis provides an introduction to some of the fixed point theorems which have proved useful in the analysis of nonlinear equations. Included is a study of Banach Contraction Principle or Banach Fixed Point Theorem, Fundamental Existence Theorem (Picard) and Schauder's Principle.

There are some examples provided in the third chapter.
CHAPTER I

CONTRACTION MAPPINGS

Definition 1. Contraction Mapping. A mapping \( f : (X, d) \rightarrow (X, d) \) is said to be a contraction if there exists a number \( k \) with \( 0 \leq k < 1 \) such that for all \( x \) and \( y \) in \( X \),

\[
(1) \quad d(f(x), f(y)) \leq kd(x, y).
\]

Notice that if \( f \) is contractive, then it is uniformly continuous.

Certain terminology is useful in studying contraction mappings. Let \( f^{(1)} = f \) and \( f^{(2)} = f \circ f \) (where \( \circ \) is the composition). Assuming \( f^{(n-1)} \) has been defined, \( f^{(n)} \) can be defined to be \( f \circ f^{(n-1)} \). \( f^{(n)} \) is called the \( n \)th iterate of \( f \). Thus if \( f : X \rightarrow X \) is defined, then the sequence \( \{ f^{(n)} \} \) of iterates of \( f \) can be defined inductively. Also, if \( x_0 \in X \), \( f^{(n)}(x_0) \) is thought of as the \( n \)th iterate of \( x_0 \) under \( f \).

Theorem 1. (Banach Contraction Principle) Suppose \( f \) is a contraction mapping and \( (X, d) \) is a complete metric space such that \( f : (X, d) \rightarrow (X, d) \). Let \( x_0 \in X \). Then the sequence \( \{ f^{(n)}(x_0) \} \) of the iterate of \( x_0 \) under \( f \) converges to a point \( z_0 \in X \) and \( f(z_0) = z_0 \). Furthermore \( z_0 \) is the
If there is a unique fixed point for \( f \).

**Proof:** The proof consists of showing the sequence \( \{ f^{(n)}(x_0) \} \) is a Cauchy Sequence in \( X \) and that its limit is the unique fixed point for \( f \).

By definition 1, there is a \( k \) such that \( 0 \leq k < 1 \) and inequality (1) is satisfied. Let \( x_i = f^{(i)}(x_0) \) for each \( i \in \mathbb{P} \). Then

\[
d(x_2, x_1) = d(f(x_1), f(x_0))
\]

By inequality (1) we have

\[
d(f(x_1), f(x_0)) \leq kd(x, x_0)
\]

Therefore,

\[
d(x_2, x_1) = d(f(x_1), f(x_0)) \leq kd(x, x_0)
\]

Hence,

\[
d(x_2, x_1) \leq kd(x, x_0)
\]

Likewise

\[
d(x_3, x_2) = d(f(x_2), f(x_1)) \leq kd(x_2, x_1) \text{ by inequality (1)}.
\]

But we have seen previously that

\[
d(x_2, x_1) = d(f(x_1), f(x_0)) \leq kd(x, x_0).
\]

Therefore we can write

\[
d(x_3, x_2) = d(f(x_2), f(x_1)) \leq kd(f(x_1), f(x_0)) = k \cdot kd(x, x_0) = k^2 d(x, x_0).
\]

Successive application of similar inequalities gives

\[
d(x_{m+1}, x_m) \leq k^md(x, x_0).
\]

This may become an endless process.
Note that if 
\[ d(x_m, x_{m-1}) \leq k^{m-1}d(x, x_0) \]
then as in above,
\[ d(x_{m+1}, x_m) = d(f(x_m), f(x_{m-1})) \]
and
\[ d(f(x_m), f(x_{m-1})) \leq kd(x_m, x_{m-1}). \]
But
\[ d(x_m, x_{m-1}) \leq k^{m-1}d(x, x_0) \]
by the assumption.
Therefore
\[ d(x_{m+1}, x_m) = d(f(x_m), f(x_{m-1})) \leq kd(x_m, x_{m-1}) \]
\[ \leq k \cdot k^{m-1}d(x, x_0) \]
Hence,
\[ d(x_{m+1}, x_m) \leq k^md(x, x_0) \]
Thus by induction, for each positive integer m,
\[ d(x_{m+1}, x_m) \leq k^md(x, x_0) \]
By using the triangle inequality
\[ d(x_{m+p}, x_m) \leq d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_m) \]
\[ \leq d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_m) \]
\[ \leq \sum_{i=1}^{p} d(x_{m+i}, x_{m+i-1}) \]
\[ \leq \sum_{i=1}^{p} k^{m+i-1}d(x, x_0) \]
\[ = k^m \cdot k^{i-1}d(x, x_0) \]
\[ = k^m \cdot \frac{1-k^p}{1-k} d(x, x_0) \]
\[ \leq \frac{k^m}{1-k} d(x, x_0) \]
Since the \( \lim (k^m) = 0 \) for \( \ell > 0 \), there exists \( m \) large enough such that
\[
d(x_{m+p}, x_m) < \ell \text{ for all } p \in P.
\]

Hence, \( \{x_m\} \) is a Cauchy Sequence in \( X \) and there is a \( \xi_0 \in X \) such that
\[
\lim \{x_m\} = \xi_0
\]

Now, the problem is to show that
\[
f(\xi_0) = \xi_0
\]

Since \( f \) is contractive, \( f \) is continuous. Therefore
\[
\lim \{f(x_m)\} = f\left(\lim \{x_m\}\right) = f(\xi_0)
\]

Now
\[
\lim \{f(x_m)\} = \lim (x_m) = f(\xi_0) = \xi_0
\]

Therefore,
\[
f(\xi_0) = \xi_0
\]

Theorem 2. Fundamental Existence Theorem (Picard). Let \( f \) be a continuous real valued function defined on an open subset \( V \) of \( \mathbb{R}^2 \). Suppose further that there exists a number \( M \) such that
\[
(1) \quad \left| f(x, y_1) - f(x, y_2) \right| \leq M \left| y_1 - y_2 \right|
\]

for all \((x, y_1)\) and \((x, y_2)\) in \( V \). If \((x_0, y_0) \in V\), then
there exists a real interval \( I = [x_0 - a, x_0 + a] \) and

a function \( y: I \to \mathbb{R} \) such that

\[
\begin{align*}
(2) & \quad y(x_0) = y_0 \\
(3) & \quad y'[x] = f(x, y(x)) \text{ for } x \in I.
\end{align*}
\]

Further \( y \) is the only function that satisfies equations (2) and (3) on the interval \( I \).

Proof: To prove this theorem it turns out to be convenient
to consider the solution of the following integral equation

for \( y \).

\[
(4) \quad y(x) = y_0 + \int_{x_0}^{x} f(t, y(t)) \, dt.
\]

A function \( y \) satisfies equations (2) and (3) if and only if it satisfies equation (4).

To check the validity of the remark above let \( y \) be the

function satisfying equations (2) and (3) then

\[
y'(t) = f(t, y(t)) \, dt \text{ for all } t \in I.
\]

Taking the integral we have

\[
\int_{x_0}^{x} y'(t) \, dt = \int_{x_0}^{x} f(t, y(t)) \, dt
\]

\[
y(t)\bigg|_{x_0}^{x} = \int_{x_0}^{x} f(t, y(t)) \, dt
\]

\[
y(x) - y(x_0) = \int_{x_0}^{x} f(t, y(t)) \, dt
\]

\[
y(x) = y(x_0) + \int_{x_0}^{x} f(t, y(t)) \, dt
\]

But equation (2) gives

\[
y(x_0) = y_0
\]
Therefore,
\[ y(x) = y_0 + \int_{x_0}^{x} f(t, y(t)) \, dt. \]
Next let \( y \) satisfy equation (4). Then
\[ y(x) = y_0 + \int_{x_0}^{x} f(x, y(x)) \, dx \text{ for } x \in I. \]
On taking the derivative of (4) we have
\[ D_x y(x) = D_x y_0 + D_x \int_{x_0}^{x} f(x, y(x)) \, dx \]
\[ y^1(x) = f(x, y(x)) \, dx \]
This is (3).
Also observe that if \( x = x_0 \), then equation (4) yields (2)
\[ y(x_0) = y_0 + \int_{x_0}^{x} f(t, y(t)) \, dt \text{ where } \int_{x_0}^{x} f(t, y(t)) \, dt = 0 \]
\[ y(x_0) = y_0 + 0 \]
\[ y(x_0) = y_0 \]
Thus the validity of the remark is verified.

The strategy will be to find a complete space \( C \) of functions and a contractive mapping \( \psi : C \rightarrow C \) that will have the following property. If there is a solution \( y \) of equation (4) then \( y \in C \) and \( y \) is a solution if and only if \( \psi(y) = y \). To accomplish this, notice that any possible solution \( y \) must be such that
\[ f(x, y(x)) \]
is defined. Now we return to the proof of the theorem.

Since \((x_0, y_0) \in V\) and \( V \) is open, there is a rectangle
\[ S = [x_0 - c, x_0 + c] \times [y_0 - b, y_0 + b] \subset V. \]
Also define
$$K = \sup \left\{ f(x,y) : (x,y) \in S \right\} + 1.$$  
Let \(a\) be a number chosen so that
(a) \(0 < a < \min \left\{ \frac{1}{m}, \frac{1}{K} \right\}\)

note that
$$[x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \subset V.$$

Let
$$C([x_0 - a, x_0 + a], \rho)$$

be the space of all functions continuous on
$$[x_0 - a, x_0 + a] = I$$

with
$$\rho(x, y) = \sup \left\{ \left| x(t) - y(t) \right| : t \in I \right\}.$$

Let \(C\) be the subset of
$$C([x_0 - a, x_0 + a], \rho)$$

consisting of all elements \(y\) such that
$$y(x) \in [y_0 - b, y_0 + b] \text{ for all } x \in [x_0 - a, x_0 + a].$$

Define
$$C = \left\{ y \in X : y[I] \subset [y_0 - b, y_0 + b] \right\}$$

Let
$$y_n \in C = \left\{ y \in X : y[I] \subset [y_0 - b, y_0 + b] \right\}$$

such that
$$y_n \xrightarrow{\rho} y.$$

We now claim \(C\) is closed. Since \(C\) is a closed subset of a complete space it is complete. Notice that if \(y \in C\), then
for all \( x \in [x_0 - a, x_0 + a] \),

\[
(b) \quad |y(x) - y_0| \leq b
\]

and

\[
(x, y(x)) \in V.
\]

Since

\[
y(x) \in [y_0 - b, y_0 + b]
\]

then

\[
y(x) \leq y_0 + b
\]

and

\[
y(x) \geq y_0 - b
\]

which implies

\[
|y(x) - y_0| \leq b.
\]

Define \( \Psi \) on \( C \) as follows:

For each \( g \in C \), let \( \Psi(g) \) be the function given by

\[
(c) \quad \Psi(g)(x) = y_0 + \int_{x_0}^{x} f(t, g(t)) \, dt
\]

The problem is to show that if \( g \in C \), then \( \Psi(g) \in C \) so that \( \Psi(C) \subset C \). From (a) if \( g \in C \), then

\[
|g(x) - y_0| \leq b \quad \text{for } x \in [x_0 - a, x_0 + a]
\]

and so \( f(x, g(x)) \) is defined for \( x \in [x_0 - a, x_0 + a] \).

Then, for all \( x \in [x_0 - a, x_0 + a] \),

\[
|\Psi(g)(x) - y_0| = \left| \int_{x_0}^{x} f(t, g(t)) \, dt \right| \quad \text{by } c.
\]

\[
\leq \int_{x_0}^{x} |f(t, g(t))| \, dt
\]

\[
\leq \int_{x_0}^{x} \text{l.u.b. } \{ |f(t, y)| : (t, y) \in S \} \, dt
\]
Then from (a) and definition of $k$, it follows that
\[ |\psi(g(x)) - y_0| \leq b \quad \text{for} \quad x \in [x_0 - a, x_0 + a] \]
Therefore,
\[ |\psi(g(x)) - y_0| \leq b \]
and the definition of the set $C$ places $\psi(g)$ in $C$. Thus, we have shown that $\psi[C] \subset C$.

The next problem is to show that $\psi: C \rightarrow C$ is a contraction.

Let
\[ I = [x_0 - a, x_0 + a] \]
Let $g_1$ and $g_2$ be elements of $C$. Then
\[
\rho(\psi(g_1), \psi(g_2)) = \text{l.u.b.}\left\{|\psi(g_1)(x) - \psi(g_2)(x)| : x \in I\right\}
\leq \text{l.u.b.}\left\{\int_{x_0}^{x} |f(t, g_1(t) - f(t, g_2(t))| \, dt : x \in I\right\}
\leq \text{l.u.b.}\left\{\int_{x_0}^{x} |g_1(t) - g_2(t)| \, dt : x \in I\right\}
\leq \text{l.u.b.}\left\{\int_{x_0}^{x} \rho(g_1, g_2) \, dt : x \in I\right\}
\]
From (a), \( ma < 1 \) and hence from the last inequality we see that \( \psi : C \rightarrow C \) is a contraction. Hence from the Banach Fixed Point Theorem or Banach Contraction Principle there is a unique \( y \) in \( C \) such that

\[ \psi(y) = y. \]

Now \( y \) is a solution for equation (2) subject to condition of equation (1). If there were another solution say \( y \), for equation (2) satisfying equation (1) and defined on the interval \( I \), then

\[ y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t)) \, dt \text{ for all } x \in I. \]

But then

\[
\left| y_1(x) - y_0 \right| \leq \left| \int_{x_0}^{x} f(t, y_1(t)) \, dt \right| \text{ for all } x \in I
\]

\[
\leq \left| \int_{x_0}^{x} k \, dt \right|
\]

\[
\leq k \left| \int_{x_0}^{x} dt \right|
\]

\[
\leq k \left| x - x_0 \right| \text{ where } \left| x - x_0 \right| < a
\]

\[
\leq ka
\]
But $k < b$ follows from (a). From the last inequality it follows that $y_1 \notin C$. Therefore,

$$|y - y_1| = 0$$

Hence,

$$y = y_1.$$  

Thus we have shown that there is an interval $I = [x_0 - a, x_0 + a]$ and a unique solution on $I$ to equations (1) and (2).

It is to be noted that in a situation covered by the previous theorem, we can find an approximate solution by first determining a suitable space $C$ and then choosing any $g_0$ in $C$ to generate the various $\psi$-interates of $g_0$. 
CHAPTER II

FIXED POINT THEOREM

We discuss the Schauder Fixed Point Theorem in this chapter. The proof of this theorem is rather difficult since it is based on some subtle details regarding the topological structures of a finite-dimensional space.

Before proving the Schauder Fixed Point Theorem we shall state several definitions and a theorem necessary for the proof:

Definition 1. S is convex if and only if for any \(x, y \in S\) \(\alpha x + \beta y \in S\) if \(\alpha + \beta = 1\).

Definition 2. The convex envelop of S is the smallest convex set containing S.

Remark: The convex envelop of S is the space of all convex linear combinations of elements of S, i.e.

\[
\sum_{i=1}^{n} \alpha_i x_i \text{ for } \sum_{i=1}^{n} \alpha_i = 1 \text{ where } x_i \in S.
\]

Proof: Let \(C = \text{Convex envelop } S\), Define

\[
\mathcal{Y} = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \sum_{i=1}^{n} \alpha_i = 1 \text{ and } x_i \in S \right\}
\]
(1) Note is \( \mathcal{U} \) convex?

Take

\[
\alpha \text{ and } \beta > \alpha + \beta = 1
\]

\[
\alpha \sum_{i=1}^{n} \alpha_i x_i + \beta \sum_{i=1}^{n} \alpha_i x_i = \sum_{i=1}^{n} \alpha_i x_i + \sum_{i=1}^{n} \beta \alpha_i x_i
\]

Now

\[
\sum_{i=1}^{n} \alpha q_i + \sum_{i=1}^{n} \beta q_i = \alpha \sum_{i=1}^{n} q_i + \beta \sum_{i=1}^{n} q_i
\]

\[
= \alpha \cdot 1 + \beta \cdot 1
\]

\[
= \alpha + \beta
\]

\[
= 1
\]

(2) \( S \subset \mathcal{U} \) and \( C \subset \mathcal{U} \) trivial we need to show \( \mathcal{Y} \subset C \). We shall prove this by induction

For \( m = 1 \) \( x_1 \in S \subset C \)

For \( m = n \) \( x = \sum_{i=1}^{n} \alpha_i x_i \subset C \sum_{i=1}^{n} \alpha_i = 1 \)

is assumed to be true. Therefore, we need to show \( n + 1 = m \)

is true.

For \( m = n + 1 \) \( y = \sum_{i=1}^{n+1} \beta_i x_i \sum_{i=1}^{n+1} \beta_i = 1 \)

Assume \( \beta_n \neq 1 \) and

\[
y = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n + \beta_{n+1} x_{n+1}
\]

\[
= (1 - \beta_{n+1}) \left( \frac{\beta_1}{n+1} x_1 + \ldots + \frac{\beta_n}{n+1} x_n \right) + \beta_{n+1} x_{n+1}
\]
Therefore

\[
\frac{\beta_1}{1-\beta_{n+1}} + \ldots + \frac{\beta_n}{1-\beta_{n+1}} = \frac{1}{1-\beta_{n+1}} (\beta_1 + \ldots + \beta_n)
\]

\[
= \frac{1-\beta_{n+1}}{1-\beta_{n+1}} = 1
\]

Definition 3. A metric space is compact if and only if for every sequence \(\{x_i\}\) in the space, there exists a subsequence converging to a point in the space.

Remark: A metric space is compact if it is complete and totally bounded. i.e. there exists a finite set \(F_\varepsilon = x_1, x_2, \ldots, x_n\) called an \(\varepsilon\) -net such that for any point \(x\) in the space, there exists \(x_i \in F_\varepsilon\) with \(\rho(x, x_i) < \varepsilon\).

Proof: Let \(X\) be a compact metric space i.e. every sequence has a convergent subsequence in \(X\). Take a Cauchy Sequence \(\{x_i\}\) in \(X\), there exists a subsequence \(x_{i_k} \rightarrow x \in X\).

We claim \(x_i \rightarrow x\).

\[
\rho(x_i, x) \leq \rho(x_i, x_{i_k}) + \rho(x_{i_k}, x)
\]

and

\[
x_{i_k} \rightarrow x
\]

Therefore \(X\) is complete.

Now let \(X\) be a compact metric space then there exists a finite number of points \(x_1, \ldots, x_n\) of \(X\) such that for each \(x \in X\)
there is an $x_k$ with $\rho(x, x_k) < \ell$.

Suppose that no such finite collection of points exists. Choose a point $y_1 \in X$. Since there is no $\varepsilon$-net $\{y_1\}$ is not an $\varepsilon$-net. There exists $y_2 \in X$ such that $\rho(y_1, y_2) > \varepsilon$.

But $\{y_1, y_2\}$ is not $\varepsilon$-net.

There exists $\rho(y_1, y_j) = \varepsilon$ if $i \neq j$.

and $\{y_1\}$ is not an $\varepsilon$-net.

Therefore, no subsequence of $\{y_j\}$ is Cauchy and no subsequence converge. Therefore $X$ is compact a contradiction.

Definition 4. A set $E \subset X$ has dimension $n$ if elements $\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_n$ exists such that (1) the differences $x_k - x_0$ $(k = 1, 2, \ldots, n)$ are linearly independent (2) every $x \in E$ can be written in the form $x = \overline{x}_0 + \sum_{k=1}^{n} a_k (\overline{x}_k - \overline{x}_0)$,

whilst an $x \in E$ exists for every $k=1, 2, \ldots, n$ such that the corresponding coefficient $a_k \neq 0$.

Definition 5. Let $x_0, x_1, x_2 \ldots, x_n$ be a system of elements in a Banach Space $X$. Suppose the differences
are linearly independent. We form the convex envelop
\( S(x_0, x_1, \ldots, x_n) \) of the elements of the system. The set
\( S(x_0, x_1, \ldots, x_n) \) is called a simplex. \( x_0, x_1, \ldots, x_n \)
are its vertices, and \( n \) is the dimension of the simplex.
Let \( S(x_0, x_1, \ldots, x_n) \) be a simplex. We choose \( k \) different
vertices \( x_{i_1}, x_{i_2}, \ldots, x_{i_k} \), and form an \((n-k)\)-dimensional
simplex from the remainder. The simplex obtained is termed
a face of \((n-k)\) dimensions of the given simplex. It lies
opposite the vertices \( x_{i_1}, x_{i_2}, \ldots, x_{i_k} \). Every element
\( x \in S, S(x_0, x_1, \ldots, x_n) \) may be written

\[
(a) \quad x = \alpha_0 x_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n
\]

where the coefficients \( \alpha_k \) are connected by the expressions:
\[
\alpha_0 + \alpha_1 + \ldots + \alpha_n = 1; \quad \alpha_k > 0 \quad (k = 0, 1, \ldots, n).
\]
We show that the representation in (a) is unique. In fact,

\[
(a) \quad x = \alpha_0 x_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n
\]

\[
(b) \quad x = x_0 + \sum_{k=1}^{n} \alpha_k (x_k - x_0)
\]

If there exists in addition to (a) an expression of the same
form, i.e.,
\[
x = \alpha'_0 x_0 + \alpha'_1 x_1 + \ldots + \alpha'_n x_n
\]
we find as above that
\[ x = x_0 + \sum_{k=1}^{n} \alpha'_k (x_k - x_0). \]

This yields, in conjunction with (b)
\[ \sum_{k=1}^{n} \alpha_k (x_k - x_0) = \sum_{k=1}^{n} \alpha'_k (x_k - x_0) \]

This is only possible, in view of the linear independence of the difference (a) where
\[ \alpha'_1 = \alpha'_1, \alpha'_2 = \alpha'_2, \ldots, \alpha'_n = \alpha'_n. \]

Definition 6. A subdivision of S will be taken to be an expression of S as the sum of a finite number of simplices of the same dimension as S defined inductively as follows: A subdivision of the one-dimensional simplex \( S(x_0, x_1) \) is taken to be the set of two simplices \( S(x_0, \frac{1}{2}(x_0 + x_1)) \) and \( S(\frac{1}{2}(x_0 + x_1), x_1) \). Suppose that a subdivision has been defined for all simplices of dimension less than \( n \), and we consider the \( n \)-dimensional face of S. Since this face is an \( (n-1) \)-dimensional simplex, a subdivision has already been defined for it. Let it consist of \( (n-1) \)-dimensional simplices \( S_1, S_2, \ldots, S_m \). If the centre \( x^* \) of the given simplex S is added to the vertices of simplices \( S_k(k=1, 2, \ldots, m) \), an \( n \)-dimensional simplex is obtained on these \( n+1 \) points. On carrying out similar constructions for each \( (n-1) \)-dimensional face of S, we obtain a subdivision of S.
Definition 7. The centre of a simplex is the point all the coordinates of which are equal.

Theorem 1. Let $X$ be an $n$-dimensional Banach Space containing a convex set $\Omega$. $P$ a continuous operation mapping the set $\Omega$ into $\Omega$ has a fixed point.

Proof: Reference [1].

After having stated the necessary definitions and theorem we can prove Schauder Fixed Point Theorem.

Theorem 2. (Schauder Fixed Point Theorem) $P$ a continuous operation mapping a set $\Omega$ of a Banach Space $X$ which is a convex closed compact set into itself has a fixed point.

Proof: Given any $\epsilon > 0$, since $\Omega$ is compact a finite $\epsilon$ -net exists in $\Omega$. Let $\epsilon$ -net consist of the elements,

$$(1) \quad x_1, x_2, \ldots, x_m$$

We now form the convex envelop $\Omega_0$ of the elements of (1). Since $\Omega$ is convex $\Omega_0 \subset \Omega$ and $\Omega_0$ has finite dimension $n \leq m - 1$. By definition (5) the set $\Omega_0$ can be written as the sum of $n$-dimensional simplices such that

(a) all the points of (1) are vertices of the simplices

(b) two simplices have either no common points or their intersection is a common face (k - dimensional $k = 0, 1, 2, \ldots n - 1$).

Now we carry out a subdivision of each of the simplices with a multiplicity large enough for the diameters of all partial simplices to be less than $\epsilon$. Let the simplices be
The set of all vertices of simplices (2) also forms an \( \varepsilon \)-net. Also the simplices (2) satisfy both conditions (a) and (b).

Now consider the operation \( P \). Since the set \( \mathcal{N} \) is compact and closed, the continuity of \( P \) implies its uniform continuity, i.e. given any \( \varepsilon > 0 \) we can find a \( \delta > 0 \) such that

\[
\| x - x^1 \| < \delta
\]

implies

\[
\| P(x) - P(x^1) \| < \varepsilon \quad (x, x^1 \in \mathcal{N}).
\]

On subdividing the simplices (2) if necessary, we can force their diameters to be less than \( \delta \). We shall suppose this has already been done, that is the diameters of all the simplices of (2) are less than \( \delta \) as well as \( \varepsilon \).

Now we construct a simplicial approximation \( P_\varepsilon \) of the operation \( P \), mapping \( \mathcal{N} \) into itself. First we define \( P_\varepsilon \) at the vertices of the simplices (2). Let \( \varepsilon \) be one such vertex. Since \( P(\varepsilon) \in \mathcal{N} \), and the vertices of the simplices (2) forms an \( \varepsilon \)-net a vertex \( \overline{\varepsilon} \) can be found such that

\[
\| \overline{\varepsilon} - P(\varepsilon) \| < \varepsilon.
\]

Now we put

\[
\overline{\varepsilon} = P_\varepsilon (\varepsilon).
\]

We now take \( x \in \mathcal{N} \) which is not a vertex of any of the simplices of (2). Let \( x \in S_k \). For the vertices of the simplex \( S_k \) we write
and write

\[ x = \sum_{i=0}^{n} \alpha_i^{(k)} x_i^{(k)}, \quad \sum_{i=0}^{n} \alpha_i^{(k)} = 1; \alpha_i^{(k)} \geq 0, \]

\[ i = 0, 1, 2, \ldots, n. \]

Put

\[ P_\xi(x) = \sum_{i=0}^{n} \alpha_i^{(k)} P_\xi(x_i^{(k)}). \]

The above definition requires no further explanation if the simplex containing \( x \) is unique. If, in addition, \( x \in S_r \) \( (r \neq k) \) we have to show \( P_\xi(x) \) does not depend on the choice of simplex. By condition (b), the intersection of the simplices \( S_k \) and \( S_r \) is a common face. Suppose that this face has vertices

\[ x_i^{(k)}, x_1^{(k)}, \ldots, x_{i_s}^{(k)} \]

of the simplex \( S_k \). On writing

\[ x_i^{(r)}, x_1^{(r)}, \ldots, x_{i_s}^{(r)} \]

for the vertices of \( S_r \), we can take

\[ (5) \ x_{i_j}^{(k)} = x_{i_j}^{(r)} \quad j = 1, 2, \ldots, s. \]

Writing \( x \) as

\[ x = \sum_{i=1}^{n} \alpha_i^{(r)} x_i^{(r)}, \quad \sum_{i=1}^{n} \alpha_i^{(r)} = 1; \alpha_i^{(r)} \geq 0 \]

\[ i = 0, 1, \ldots, n. \]
Since
\[ a_i^{(k)} = a_i^{(r)} = 0 \quad ij \neq ji \quad j=1, 2, \ldots, S \]

By (5) we have
\[ \sum_{j=0}^{s} a_{ij}^{(k)} x_{ij}^{(k)} = \sum_{j=0}^{s} a_{ij}^{(r)} x_{ij}^{(r)} \]
whence it follows that
\[ a_{ij}^{(k)} = a_{ij}^{(r)} \quad j = 0, 1, 2, \ldots, S \]

Therefore, if we define \( p_{\epsilon} (x) \) by starting from the simplex \( S_\tau \), we have by (6), (5) and (7),
\[ p_{\epsilon} (x) = \sum_{i=0}^{n} a_{ij}^{(r)} p_{\epsilon} (x_{ij}^{(r)}) = \sum_{j=0}^{s} a_{ij}^{(r)} p_{\epsilon} (x_{ij}^{(r)}) \]
\[ = \sum_{j=0}^{s} a_{ij}^{(k)} p_{\epsilon} (x_{ij}^{(k)}) \]
\[ = \sum_{i=0}^{n} a_{ij}^{(k)} p_{\epsilon} (x_{ij}^{(k)}). \]

The conditions of Theorem (1) are, therefore, satisfied for the operation \( p_{\epsilon} \). It follows from this theorem that a fixed point \( x_{\epsilon} \in \cap_0 \) of \( p_{\epsilon} \) exists, i.e.,
\[ x_{\epsilon} = p_{\epsilon} (x_{\epsilon}) \]
Let \( z_0, z_1, \ldots, z_n \) be the vertices of the simplex of (2)
to which the point $x_\varepsilon$ belongs. Since
\[ \|z_i - z_j\| < \delta, \]
we have by (3)
\[ (8) \quad \|P(z_i) - P(z_j)\| < \varepsilon \quad i, j = 0, 1, \ldots, n. \]

If we recall that by definition
\[ (9) \quad \|P(z_i) - P(z_j)\| < 3\varepsilon \quad i, j = 0, 1, \ldots, n. \]
Now writing $x_\varepsilon$ as
\[ x = \sum_{i=0}^{n} a'_i (\varepsilon) z_i, \quad \sum_{i=0}^{n} a'_i (\varepsilon) = 1; a'_i (\varepsilon) = 0, i = 0, 1, \ldots, n. \]
By the definition of $P_\varepsilon$ we have
\[ x_\varepsilon = P_\varepsilon (x_\varepsilon) = \sum_{i=0}^{n} a'_i (\varepsilon) P_\varepsilon (z_i). \]

On taking (9) into account, from this we obtain
\[ (10) \quad \|x_\varepsilon - P(z_j)\| = \left\| \sum_{i=0}^{n} a'_i (\varepsilon) [P_\varepsilon (z_i) - P_\varepsilon (z_j)] \right\| \]
\[ \leq 3\varepsilon \sum_{i=0}^{n} a'_i (\varepsilon) = 3\varepsilon \quad j = 0, 1, \ldots, n. \]
\[ \|x_\varepsilon - z_i\| < \delta, \quad i = 0, 1, \ldots, n \]
and (3) give
\[ (11) \quad \|P(x_\varepsilon) - P(z_i)\| < \varepsilon \quad i = 0, 1, \ldots, n. \]

We now combine (10), (8) and (11) to obtain
Given any $\varepsilon > 0$, a point $x_\varepsilon \in \mathcal{O}$, therefore, exists such that

$$\|x_\varepsilon - P(x_\varepsilon)\| \leq 5\varepsilon.$$

Taking a sequence $\varepsilon_k \rightarrow 0$ and construct the point $x_k = x_\varepsilon$ for every $k = 1, 2, \ldots$. We can assume without loss of generality, since $\mathcal{O}$ is compact and closed, that

$$x_k \longrightarrow x^* \in \mathcal{O}.$$  

$k \rightarrow \infty$

But now,

$$\|x^* - P(x^*)\| \leq \|x^* - x_k\| + \|x_k - P(x_k)\| + \|P(x_k) - P(x^*)\|$$

and all the terms on the right hand side tend to zero (the last by virtue of the continuity of $P$). Therefore,

$$X^* = P(x^*)$$

which proves the existence of the fixed point.
CHAPTER III

APPLICATIONS

Example 1:

\[ f(x) = \ln (1+e^x) \] and \( f: \mathbb{R} \to \mathbb{R} \)

where \( X = \mathbb{R} \) is a complete metric space.

Remark: The condition that \( T \) be a strict contraction in Banach's Theorem may be replaced in general by the weather condition.

(1) \( \rho(Tx, Ty) < \rho(x, y), \ x \neq y. \)

Define

\[ \rho(x, y) = |x-y| \]

\[ f(x) - f(y) = f'(c) (x-y), \quad x < c < y \]

Since \( f \) is differentiable

\[ \rho(f(x), f(y)) = |f(x) - f(y)| = |f'(c)||x-y| \]

\[ = \frac{e^c}{1+e^c} \rho(x, y) < \rho(x, y) \]

Thus \( f \) satisfies (1).

\( f \) does not have a fixed point. For if it did

\[ \ln (1+e^x) = x \]

\[ 1+e^x = e^x \] which is a contradiction

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Therefore $f$ is not a strict contraction satisfying (1).

Example 2:

Suppose $(X, d)$ is a complete metric space and $(S, d)$ is a subspace of $(X, d)$. Let $f$ be a mapping from $S$ onto $X$ such that for an $\alpha > 1$,

$$d(f(x), f(y)) \geq \alpha d(x, y)$$

Then, $f^{-1}$ is a contraction mapping.

Solution: The problem here is to show $f$ is a contraction, i.e. $d(f^{-1}(x), f^{-1}(y)) \leq kd(x, y)$.

Let

$$x = f^{-1}(\rho_1)$$

and

$$y = f^{-1}(\rho_2)$$

$$d(f(x), f(y)) \geq \alpha d(x, y) \text{ for } 1$$

$$d(x, y) \leq \frac{1}{\alpha} d(f(x), f(y)) \text{ for } 0 \leq k = \frac{1}{\alpha} \leq 1$$

$$d(f^{-1}(\rho_1), f^{-1}(\rho_2)) \leq kd(\rho_1, \rho_2)$$

Therefore $f^{-1}$ is a contraction on $(s, d)$. 
Example 3:

Prove that if $f : [a, b] \to [c, b]$ is differentiable and $|f'(x)| \leq k < 1$ for $x \in [a, b]$, then $f$ is a contraction mapping on $[a, b]$.

Solution: Define

$$d(b, a) = |b - a|$$

Since $f$ is differentiable by mean value theorem

$$f(b) - f(a) = f'(x)(b - a)$$

$$d(f(b), f(a)) = |f(b) - f(a)| = |f'(x)||b - a|$$

where $|f'(x)| \leq k < 1$

So

$$d(f(b), f(a)) \leq kd(b, a)$$

Therefore $f$ is a contraction mapping.
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