Mean number of real zeros of a random trigonometric polynomial

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ABSTRACT
MATHEMATICAL SCIENCES

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MEAN NUMBER OF REAL ZEROS OF A RANDOM TRIGONOMETRIC POLYNOMIAL

Advisor: Dr. J. Ernest Wilkins, Jr.

Thesis dated March, 1995

My interest is in estimating the average number of real zeros of the trigonometric cosine polynomial,

\[ a_1 \cos x + 2^{1/2} a_2 \cos 2x + n^{1/2} a_n \cos nx, \] (1)

when the coefficients \( a_j \) are independent normally distributed random variables with mean 0 and variance 1. The same problem without the square roots in the coefficients has been studied by Das [Proc. Cambridge Philos. Soc. 64 (1968), 721-729], and by Wilkins [Proc. Am. Math. Soc. 111 (1991), 851-863]. I generalize their results and find an asymptotic expansion for the average number \( \nu_n \) of zeros of the polynomial (1) on the interval \((0,2\pi)\). This expansion will have the form,

\[ \nu_n = 2^{-1/2} (2n + 1) \left[ 1 + E_1 (2n + 1)^{-1} + E_2 (2n + 1)^{-2} + E_3 (2n + 1)^{-3} + O((2n + 1)^{-4}) \right], \]

in which the coefficients \( E_1, E_2 \) and \( E_3 \) are numerical constants whose values are \(-0.378124\), \(-1/2\) and \(0.5523\) respectively.
MEAN NUMBER OF REAL ZEROS OF A
RANDOM TRIGONOMETRIC POLYNOMIAL

A THESIS
SUBMITTED TO THE FACULTY OF CLARK ATLANTA UNIVERSITY
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CHAPTER 1
INTRODUCTION

Suppose that \( a_1, a_2, \ldots, a_n \) are independent, normally distributed random variables with mean 0 and variance 1, and that \( v_n \) is the mean value of the number of zeros on the interval \((0, 2\pi)\) of the trigonometric polynomial \( a_1 \cos x + 2^{\sigma}a_2 \cos 2x + \ldots + n^{\sigma}a_n \cos nx \). Das [3] has shown that

\[
v_n = \{(2\sigma + 1)/(2\sigma + 3)\}^{\frac{1}{3}}(2n) + O(n^{\frac{1}{6}})
\]

for large \( n \). Christensen and Sambandham [2] (cited by Bharucha-Reid and Sambandham [1]) have shown that

\[
|v_n - (2n/3^{\frac{1}{6}})| \leq 10n^{\frac{1}{6}} \text{ when } n \geq 25.
\]

Wilkins [5] has shown that

\[
v_n = 3^{-\frac{1}{6}}(2n + 1)\sum_{r=0}^{3} (2n + 1)^{-r}D_r + O(2n + 1)^{-3},
\]

in which \( D_0 = 1, D_1 = 0.232423, D_2 = -0.25973 \) and \( D_3 = 0.2172 \). Wilkins [6] has also shown that, when \( \sigma \) is a positive integer \( p \),

\[
v_n = (2n + 1)\mu_p\sum_{r=0}^{3} (2n + 1)^{-r}D_{rp} + O(2n + 1)^{-3},
\]

in which \( D_{rp} \) are explicitly defined constants \((D_{0p} = 1)\) and \( \mu_p = ((2p + 1)/(2p + 3))^{\frac{1}{6}} \). In this paper we will consider the special case for which \( \sigma = \frac{V}{2} \). More precisely, we will prove the following theorem (in which we have dropped the subscript \( \frac{1}{6} \)).
Theorem. For large $n$, it is true that

\begin{equation}
\nu_n = 2^{-\frac{n}{2}}(2n + 1) \left[ E_0 + E_1 (2n + 1)^{-1} + E_2 (2n + 1)^{-2} + E_3 (2n + 1)^{-3} \right] + O((2n + 1)^{-3}), \tag{1.1}
\end{equation}

in which $E_0 = 1$, $E_2 = -\frac{1}{2}$, and $E_1$ and $E_3$ are explicitly defined constants (see Equations (3.37), and (3.39)) whose numerical values are approximately $-0.378124$ and $0.5523$, respectively.

After a statement of the basic formulas on which our analysis rests, we devote Chapter 2 to the derivation of a convergent series representation for $\nu_n$ (cf. Lemma 7). Asymptotic representations for the first few coefficients in that series are derived in Chapter 3 and are used to establish the Theorem. We have calculated and recorded in Table 1, 6D values of $\nu_n$ when $n = 2(1)60$. In Chapter 4 we use these values to assess the accuracy of (1.1). In particular, the approximation to $\nu_n$ obtained from (1.1) if the $O((2n + 1)^{-3})$ term is neglected produces 5D values that differ from the calculated values by at most $10^{-5}$ when $n \geq 8$ and by only about 0.1% when $n$ is as small as 2.
CHAPTER 2
PRELIMINARY ANALYSIS

It is known [1, p. 107] that, when \( n \geq 2 \),

\[
\nu_n = \pi^{-1} \int_0^{2\pi} F_n(x) \, dx,
\]

in which

\[
F_n = A_n^{-1} (A_n C_n - B_n^2)^{1/4},
\]

\[
A_n = \sum_{j=1}^{n} j \cos^2 jx,
\]

\[
B_n = \sum_{j=1}^{n} j^2 \cos jx \sin jx,
\]

\[
C_n = \sum_{j=1}^{n} j^3 \sin^2 jx.
\]

It is then obvious that

\[
\nu_n = 4\pi^{-1} \int_0^{\pi/2} F_n(x) \, dx.
\]

We will need the explicit representations for \( A_n, B_n \) and \( C_n \) stated in the following lemma.

**Lemma 1.** It is true that

\[
8A_n = (2n + 1)^2 g_0 + (2n + 1) g_1 + g_2,
\]

\[
16B_n = (2n + 1)^3 h_0 + (2n + 1)^2 h_1 + (2n + 1) h_2 + h_3,
\]

\[
32C_n = (2n + 1)^4 k_0 + (2n + 1)^3 k_1 + (2n + 1)^2 k_2
\]

\[
+ (2n + 1) k_3 + k_4,
\]

if the quantities \( g_0, g_1, g_2, h_0, h_1, h_2, h_3, k_0, k_1, k_2, k_3 \) and \( k_4 \) are defined so that

\[
g_0 = \frac{1}{2} + z^{-1} \sin z - z^{-2}(1 - \cos z),
\]

\[
\]
\[(2.10)\] \(g_1 = f(x) \sin z, \quad f(x) = \csc x - x^{-1},\)

\[(2.11)\] \(g_2 = -(\frac{1}{2} + \varphi(x) + f'(x) \cos z), \quad \varphi(x) = \csc^2 x - x^{-2},\)

\[(2.12)\] \(h_0 = -z^{-1} \cos z + 2z^{-2} \sin z - 2z^{-3}(1 - \cos z),\)

\[(2.13)\] \(h_1 = -f(x) \cos z,\)

\[(2.14)\] \(h_2 = -2f'(x) \sin z,\)

\[(2.15)\] \(h_3 = \varphi'(x) + f''(x) \cos z,\)

\[(2.16)\] \(k_0 = \frac{3}{4} - z^{-1} \sin z - 3z^{-2} \cos z + 6z^{-3} \sin z \)

\[-6z^{-4}(1 - \cos z),\]

\[(2.17)\] \(k_1 = -f(x) \sin z,\)

\[(2.18)\] \(k_2 = 3f'(x) \cos z - \frac{1}{2},\)

\[(2.19)\] \(k_3 = 3f''(x) \sin z,\)

\[(2.20)\] \(k_4 = -\varphi''(x) - f'''(x) \cos + \frac{1}{4},\)

\[(2.21)\] \(z = (2n + 1)x.\)

It is a consequence of a known result [4, p. 133, Eq. (31)] (or of a simple mathematical induction) that

\[(2.22)\] \(B_n = 2n(n + 1) + (2n + 1) \csc x \sin z \)

\[-(1 - \cos x \cos z) \csc^2 x.\)

The validity of (2.6), together with (2.9), (2.10) and (2.11), is now a consequence of some algebraic manipulation. The validity of (2.7) together with (2.12), (2.13), (2.14) and (2.15), and of (2.8) together with (2.16), (2.17), (2.18), (2.19) and (2.20), follows at once from (2.6) and the observations that

\[2B_n = -dA_n/dx,\]

\[(2.23)\] \(32C_n = 16 \sum_{j=1}^{n} j^3 - 16dB_n/dx\)
\[ (2n + 1)^4/4 - (2n + 1)^2/2 + \sqrt{4} - 16dB_n/dx. \]

For future reference we record in the following lemma the power series expansions of the functions \( f(x) \) [5, Lemma 2] and \( \varphi(x) \).

**Lemma 2.** If \( \beta_{2m} \) is the Bernoulli number of order \( 2m \), then

\[
\begin{align*}
    f(x) &= \sum_{m=1}^{\infty} (-1)^{m-1}(2^{2m} - 2) \beta_{2m}x^{2m-1}/(2m)! \\
    &= (x/6) + (7x^3)/360 + (3x^5/15120) + \ldots,
\end{align*}
\]

(2.24)

\[
\varphi(x) = \csc^2 x - x^{-2} = f^2(x) + 2x^{-1}f(x)
\]

\[
= (1/3) + (x^2/15) + (x^4/630) + \ldots.
\]

The power series in (2.24) converge when \(|x| < \pi\), and their coefficients are positive.

We calculate the first factor \( A_n^{-1} \) in (2.1) in the following:

**Lemma 3.** There exists an integer \( n_0 \) such that, if

\[
0 \leq x \leq \pi/2 \quad \text{and} \quad n \geq n_0,
\]

(2.25)

\[
(8A_n)^{-1} = (2n + 1)^{-2}g_0^{-1}\sum_{r=0}^{\infty} (2n + 1)^{-r}b_r,
\]

in which \( b_0 = 1, b_1 = -g/g_0, b_2 = (g_1/g_0)^2 - g/g_0, b_3 = 2(g_1g_2/g_0^2) - g_1^3/g_0^3, \ldots. \) The series in (2.25) converges absolutely and uniformly when \( 0 \leq x \leq \pi/2, n \geq n_0. \)

It is a consequence of (2.9) that

(2.26)

\[
g_0(z) = \int_0^1 t(1 + \cos zt)dt.
\]

Therefore, \( g_0(z) > 0 \) when \( 0 \leq z < \infty \), and \( \lim_{z \to \infty} g_0(z) = \frac{1}{2} > 0. \)
We infer that both \( g_0 \) and \( g_0^{-1} \) are bounded functions of \( z \). Because the functions \( g_1 \) and \( g_2 \) defined in (2.10) and (2.11) are obviously bounded (uniformly in \( n \) and \( x \) when \( n \geq 2 \) and \( 0 \leq x \leq \pi/2 \)), it follows that there is a positive integer \( n_0 \) so large that

\[
(2.27) \quad (2n + 1)^{-1}|g_1/g_0| + (2n + 1)^{-2}|g_2/g_0| < 0.95
\]

when \( n \geq n_0 \) and \( 0 \leq x \leq \pi/2 \). The expression (2.6) can be inverted for such values of \( n \) and \( x \); this yields (2.25), in which the first four coefficients are those specified in Lemma 3.

A straightforward calculation, based on (2.6) through (2.8), shows the validity of the following lemma.

**Lemma 4.** It is true that

\[
(2.28) \quad 256(A_nC_n - B_n^2) = (2n + 1)^6f_0 + (2n + 1)^5f_1 + (2n + 1)^4f_2
\]

\[
+ (2n + 1)^3f_3 + (2n + 1)^2f_4 + (2n + 1)f_5 + f_6,
\]

if the quantities \( f_r \) are defined so that

\[
(2.29) \quad f_r = \sum_{m=0}^{r} (g_m k_r - h_m h_r).
\]

(It is understood that \( g_m = 0 \) if \( m > 2 \), \( h_m = 0 \) if \( m > 3 \), and \( k_m = 0 \) if \( m > 4 \).) In particular,

\[
(2.30) \quad f_0 = g_0 k_0 - h_0^2
\]

\[
= (1/8) - (4z)^{-1}\sin z - (5/4)z^{-2}(1 - \cos z)
\]

\[
+ 4z^{-3}\sin z - z^{-4}(1 - \cos z)^2
\]

\[
- 4z^{-5}\sin z(1 - \cos z) + 2z^{-6}(1 - \cos z)^2,
\]

\[
(3.31) \quad f_1 = g_0 k_1 + g_1 k_0 - 2h_0 h_1
\]

\[
= \{-4^{-1}\sin z - 2z^{-1} + z^{-2}\sin z
\]
\[ f_2 = g_0k_2 + g_1k_1 + g_2k_0 - 2h_0h_2 - h_1^2 \]
\[ = \left\{ \left( \frac{5}{4} \right) \cos z + z^{-2}(6 \cos^2 z + 8 \sin^2 z - 3 \cos z + z^{-3}(2\sin z \cos z - 8 \sin z) \right. \right. \]
\[ \left. \left. - 6z^{-4}(\cos^2 z - \cos z) \right\} f'(x) \right. \]
\[ + \left. \left( - \frac{3}{4} + z^{-1}\sin z + 3z^{-2}\cos z - 6z^{-3}\sin z \right. \right. \]
\[ \left. \left. + 6z^{-4}(1 - \cos z) \right\} \phi(x) \right. \]
\[ - f^2(x) + \left( -3/8 + z^{-2}\cos z + 2z^{-3} - 3z^{-3}\sin z \right. \]
\[ \left. + 3z^{-4}(1 - \cos z) \right\} \]

\[ (2.32) \]
\[ f_3 = g_0k_3 + g_1k_2 + g_2k_1 - 2h_0h_3 - 2h_1h_2 \]
\[ = \left\{ \left( \frac{3}{2} \right) \sin z + z^{-1}(3 \sin^2 z + 2 \cos^2 z) \right. \]
\[ - z^{-2}(3 + \cos z) \sin z + 4z^{-3}(1 - \cos z) \cos z \right\} f''(x) \]
\[ + \left( 2z^{-1}\cos z - 4z^{-2}\sin z + 4z^{-3}(1 - \cos z) \right\} \phi'(x) \]
\[ + f(x) \phi(x) \sin z. \]

We calculate the second factor \((A_nC_n - B_n^2)^{\frac{1}{4}}\) in \((2.1)\) in the following lemma:

**Lemma 5.** There exists an integer \(n_0\) such that \(n_0 \geq 2\) and

\[ (2.34) \quad 16(A_nC_n - B_n^2)^{\frac{1}{4}} = (2n + 1)^3 f_0^3 \sum_{r=0}^{n} (2n + 1)^{-r} c_r \]

when \(n \geq n_0\), in which

\[ (2.35) \quad c_0 = 1, \quad c_1 = f_2/2f_0, \quad c_2 = f_2/2f_0 - f_1^2/8f_0^2, \]
\[ c_3 = f_3/2f_0 - f_1f_2/4f_0 + f_1^3/16f_0^3, \cdots. \]

The series in \((2.34)\) converges absolutely and uniformly when \(0 \leq x \leq \pi/2\) and \(n \geq n_0\).
We infer from (2.12) and (2.16) that
\[ (2.36) \quad h_0 = \int_0^1 t^2 \sin zt \, dt, \quad k_0 = \int_0^1 t^3 (1 - \cos zt) \, dt, \]
so that (2.26) and the Schwarz inequality imply that
\[ f_0 = g_0 k_0 - h_0^2 > 0 \quad \text{when} \quad 0 < z < \infty. \]
Moreover, \( f_0 = z^2/48 + O(z^4) \)
for small \( z \) and \( f_0 = 1/8 + O(z^{-1}) \) for large \( z \).

Because \( f \) is an odd function of \( x \), \( \varphi \) is an even function of \( x \), and \( 0 \leq x \leq \pi/2, \) \( 0 \leq x = z/(2n + 1) \leq z/5 \) when \( n \geq 2 \), it follows from (2.36), (2.12), (2.13), (2.14) and (2.15) that \( h_x = O(z) \), uniformly in \( n \) and \( x \), and from (2.36) and (2.16) through (2.20) that \( k_x = O(z^2) \), uniformly in \( n \) and \( x \). It is now a consequence of (2.29), and the earlier observation that \( g_x = O(1) \), that \( f_x = O(z^2) \), uniformly in \( n \) and \( x \), and \( f_x = O(1) \) uniformly in \( n \) and \( x \).

Because it is obvious that \( h_x = O(1), \) \( k_x = O(1) \), uniformly in \( n \) and \( x \), we conclude that \( f_x/f_0 = O(1) \), uniformly in \( n \) and \( x \). We can accordingly choose \( n_0 \) so large that (2.27) holds and
\[ (2.37) \quad \sum_{r=1}^{6} (2n + 1)^{-r} |f_x/f_0| < 0.95 \]
when \( n \geq n_0 \) and \( 0 \leq x \leq \pi/2 \). Lemma 5 now follows upon extracting the square root of the expression in (2.28).

If we use (2.1), (2.5) and Lemmas 3 and 5, we obtain the following lemmas.

**Lemma 6.** It is true that, when \( n \geq n_0 \),
\[ (2.38) \quad 2F_n(x) = (2n + 1)^{\frac{1}{2}} g_0^{-1} \sum_{r=0}^{\infty} (2n + 1)^{-r} u_r, \]
in which
The series (2.38) converges absolutely and uniformly when \(0 \leq x \leq \pi/2\) and \(n \geq n_0\).

Lemma 7. It is true when \(n \geq n_0\) that

\[
(2.40) \quad v_n = (2n + 1) \sum_{r=0}^{\infty} (2n + 1)^{-r} v_r,
\]

in which

\[
(2.41) \quad v_r = 2\pi^{-1} \int_0^{\pi/2} G(z) u_x \, dx,
\]

\[
(2.42) \quad G(z) = f_{\psi}^b g_0^{-1}.
\]

We observe that \(G(z) = O(1)\) and \(G(z) = O(z)\), both uniformly in \(n\) and \(x\), and that the series (2.40) converges absolutely and uniformly in \(n\) when \(n \geq n_0\).
CHAPTER 3
PROOF OF THE THEOREM

In the next four lemmas we will exhibit constants $S_{rm}(0 \leq r + m \leq 3)$ and $S_{r}(r = 0,1,2,3)$ such that

$$2^{3r}v_{r} = \sum_{m=0}^{3-r} (2n + 1)^{-m}S_{rm} + (-1)^{n}(2n + 1)^{r-3}S_{r} + O((2n + 1)^{r-4})$$

when $r = 0,1,2,3$. In the proofs of these lemmas it will be convenient to use $T_{q}(z)$ as a generic symbol for a trigonometric sine polynomial of degree $q$, not necessarily the same at each occurrence.

Lemma 8. Eq. (3.1) is true when $r = 0$ if

$$S_{00} = 1, \quad S_{01} = 2\pi^{-1}\int_{0}^{\pi}(2^{n}G(z) - 1)dz,$$

$$S_{02} = \pi^{-2}, \quad S_{03} = 0, \quad S_{0} = -32\pi^{-3}.$$

If $z$ is large we deduce from the definitions (2.41), (2.9) and (2.33) of $G(z), g_{0}$ and $f_{0}$ that

$$2^{n}G(z) = 1 - 3z^{-1}\sin z - (2z)^{-2}(11\cos 2z + 28\cos z + 1) + z^{-3}T_{3}(z) + O(z^{-4}).$$

It then follows from (2.41) and (2.39) when $r = 0$, and Lemmas 3 and 5, that

$$2^{3n}v_{0} = 1 + \lambda^{-1}\int_{0}^{\pi}(2^{n}G(z) - 1)dz - \lambda^{-1}\int_{0}^{\pi}(2^{n}G(z) - 1)dz,$$

in which

$$\lambda = (2n + 1)\pi/2.$$

If the last term in (3.4) is evaluated with the help of
(3.3) and some integrations by parts, we find that Lemma 8 is true.

**Lemma 9.** Eq. (3.1) is true when \( r = 1 \) if

\[
S_{10} = 0, \quad S_{11} = -(5/\pi)\int_0^{\pi/2} x^{-1}f(x)\,dx, \quad S_1 = -24/\pi^3,
\]
\[
S_{12} = -(1/3\pi)\int_0^{\pi/2} z(2^{H}H(z) - 3 \sin z)
- (2z)^{-1}(5 + 11 \cos 2z)\,dz
\]

in which

\[
H(z) = [(2z)^{-1}(g_0 - k_0) + g_0^{-1})\sin z - f_0^{-1}h_0 \cos z]G(z).
\]

If \( H(z) \) is the function defined in (3.7), we infer from (2.39) and Lemmas 3, 4 and 5 that

\[
G(z)u_1 = -2f(x)H(z).
\]

If \( z \) is large we deduce from the definitions (2.9), (2.12), (2.16) and (2.30) of \( g_0, h_0, k_0 \) and \( f_0 \), respectively, and from (3.3) that

\[
H(z) = H^*(z) + O(z^{-3}),
\]
\[
2^H H^*(z) = 3 \sin z + (2z)^{-1}(5 + 11 \cos 2z)
+ (8z^2)^{-1}(151 \sin z - 60 \sin 2z - 69 \sin 3z).
\]

Moreover, \( H(z) = O(1) \) and \( H^*(z) = O(z^{-1}) \) for small \( z \). Hence (3.9) is true for all positive \( z \). It now follows from (2.41) and (3.8) that

\[
-\pi v_{1/2}^2 = I_1 + I_2 + I_3,
\]

in which

\[
I_1 = \int_0^{\pi/2} 2^{1/2} \{H(z) - H^*(z)\}\{f(x) - (x/6)\} \,dx,
\]
\[
I_2 = (2^{1/2}/6)\int_0^{\pi/2} x\{H(z) - H^*(z)\} \,dx,
\]
\[
I_3 = \int_0^{\pi/2} 2^{1/2}H^*(z)f(x) \,dx.
\]
We conclude from (2.24), (3.9) and (3.12) that

\[(3.15) \quad I_1 = \int_0^{\pi/2} O(x^{-3}) O(z^{-3}) \, dx = O\left((2n + 1)^{-3}\right).\]

Similarly, we conclude from (2.21), (3.9), (3.13) and (3.5) that

\[(3.16) \quad 6I_2 = (2n + 1)^{-3} 2^{\frac{1}{2}} \int_0^1 z \{H(z) - H^*(z)\} \, dz\]

\[= (2n + 1)^{-3} 2^{\frac{1}{2}} \int_0^1 z \{H(z) - H^*(z)\} \, dz\]

\[ - 6 (2n + 1)^{-3} 2^{\frac{1}{2}} \int_0^1 O(z^{-3}) \, dz\]

\[= (2n + 1)^{-3} 2^{\frac{1}{2}} \int_0^1 z \{H(z) - H^*(z)\} \, dz + O((2n + 1)^{-3}).\]

In order to evaluate the integral (3.14) for \(I_3\), we observe that

\[(3.17) \quad \int_0^{\pi/2} f(x) \sin z \, dx = \left[ - (2n + 1)^{-1} f(x) \cos z \right.\]

\[+ (2n + 1)^{-2} f'(x) \sin z + (2n + 1)^{-3} f''(x) \cos z \left.\right|_0^{\pi/2}\]

\[ - (2n + 1)^{-3} \int_0^{\pi/2} f'''(x) \cos z \, dx\]

\[= (2n + 1)^{-2} \left(\frac{4}{\pi^2}\right) (-1)^n + O((2n + 1)^{-3}),\]

because \(f'\left(\pi/2\right) = 4/\pi^2\);

\[(3.18) \quad \int_0^{\pi/2} z^{-1} f(x) \, dx = (2n + 1)^{-1} \int_0^{\pi/2} x^{-1} f(x) \, dx;\]

\[(3.19) \quad 4 \int_0^{\pi/2} z^{-1} f(x) \cos 2z \, dx = \left[ 2 (2n + 1)^{-2} x^{-1} f(x) \sin 2z \right.\]

\[+ (2n + 1)^{-3} (x^{-1} f(x))' \cos 2z \left.\right|_0^{\pi/2}\]

\[ - (2n + 1)^{-3} \int_0^{\pi/2} (x^{-1} f(x))'' \cos 2z \, dx\]

\[= O\left((2n + 1)^{-3}\right);\]

\[(3.20) \quad \int_0^{\pi/2} z^{-2} f(x) \sin qz \, dx = (2n + 1)^{-2} f'(0) \int_0^1 z^{-1} \sin qz \, dz\]

\[+ (2n + 1)^{-2} \int_0^{\pi/2} x^{-2} \{f(x) - xf'(0)\} \sin qz \, dx\]

\[= (2n + 1)^{-2} (\pi/12)\]

12
\[-(2n+1)^{-3}q^{-1}[x^{-2}(f(x) - xf'(0)) \cos qz]_0^{\pi/2}\]
\[+ (2n+1)^{-3}q^{-1}\int_0^{\pi/2} [x^{-2}(f(x) - xf'(0))]' \cos qz \, dx\]
\[+ \mathcal{O}(2n+1)^{-3}\]
\[= (2n+1)^{-2}(\pi/12) + \mathcal{O}(2n+1)^{-3},\]
because \(f'(0) = 1/6\) and \(f(x) = xf'(0) + O(x^3)\).

If we use the known identity,
\[(3.21) \int_0^{\pi} z^{-1} \sin qz \, dq = \pi/2 \quad (q > 0),\]
and (3.14), (3.10), (3.17), (3.18), (3.19) and (3.20) to evaluate \(I_3\), and then combine that result with (3.15), (3.16) and (3.11) to evaluate \(v_1\), we conclude that Lemma 9 is true.

**Lemma 10.** Eq. (3.1) is true when \(r = 2\) if
\[(3.22) S_{20} = (2/\pi) \int_0^{\pi/2} \phi(x) \, dx - (1/2) - (5/2\pi) \int_0^{\pi/2} f^2(x) \, dx,\]
\[S_{21} = (3\pi^{-1}) \int_0^{\pi/2} [2^k J(z) + 1 - 7 \cos z] \, dz, \quad S_2 = 56/\pi^3,\]
in which
\[(3.23) J(z) = [6g_0^{-1} - 3f_0^{-1}k_0 + 2f_0^{-1}h_0 \sin z]
+ (- g_0^{-1} + (k_0 - 3g_0_0) (2f_0)^{-1} (1 - \cos z)) G(z).\]

We infer from the lemmas in section 2 that
\[(3.24) G(z) u_2 = k(z) \{f'(x) - f'(0)\} + M(z) \{\phi(x) - \phi(0)\}
-L(z) f^2(x) + f'(0) J(z),\]
in which \(J(z)\) is defined in (3.23), and \(K(z), M(z)\) and \(L(z)\) are defined so that
\[K/G = ((2f_0g_0)^{-1}(3g_0^2 - g_0k_0 + 2f_0)) \cos z + 2f_0^{-1}h_0 \sin z,\]
\[M/G = (2f_0g_0)^{-1}(2f_0 - k_0g_0),\]
\[
L/G = [(16f_0^2)^{-1}(g_0 - k_0)^2 - (4f_0^2)^{-1}h_0^2 \\
- (4f_0g_0)^{-1}(g_0 - k_0) - (2g_0^2)^{-1}] (1 - \cos 2z) \\
+ (g_0^{-1} - (2f_0)^{-1}(g_0 - k_0)) ((2f_0)^{-1}h_0 \sin 2z) \\
+ ((2f_0^2)^{-1}h_0^2 + (2f_0)^{-1}).
\]

We need to know that, for all positive \( z \),
\[
(3.25) \quad 2^\#K(z) = 7 \cos z - (2z)^{-1}15 \sin 2z + O(z^{-2}),
\]
\[
(3.26) \quad 2^\#M(z) = 1 - 2z^{-1} \sin z + O(z^{-2}),
\]
\[
(3.27) \quad 2^\#L(z) = (10 + 22 \cos 2z \\
+ z^{-1} (111 \sin z - 69 \sin 3z))/8 + O(z^{-2}).
\]

Moreover, if \( F(x) \) is either of the functions \( f'(x) - f'(0) \)
or \( f^2(x) \), so that \( F(x) = O(x^2) \) when \( 0 \leq x \leq \pi/2 \), we see that
\[
\int_0^{\pi/2} F(x) z^{-1} T_2(z) \, dx = O((2n + 1)^{-2}),
\]
\[
\int_0^{\pi/2} F(x) \, O(z^{-2}) \, dx = O((2n + 1)^{-2}).
\]

Therefore,
\[
(3.28) \quad 2^\# \int_0^{\pi/2} K(z) \{f'(x) - f'(0)\} \, dx \\
= (2n + 1)^{-1} 7 \{f'(\pi/2) - f'(0)\}(\pi)^n + O((2n + 1)^{-2}),
\]
\[
(3.29) \quad 2^\# \int_0^{\pi/2} M(z) \{\varphi(x) - \varphi(0)\} \, dx \\
= \int_0^{\pi/2} \varphi(x) \, dx - \pi/6 - O((2n + 1)^{-2}),
\]
\[
(3.30) \quad 2^\# \int_0^{\pi/2} L(z) f^2(x) \, dx = 5/4 \int_0^{\pi/2} f^2(x) \, dx + O((2n + 1)^{-2}).
\]

We also need to know that, for large \( z \),
\[
(3.31) \quad J(z) = J^*(z) + z^{-1}T_2(z) + O(z^{-2}),
\]
\[
(3.32) \quad 2^\#J^*(z) = -1 + 7 \cos z.
\]

It now follows, by reasoning like that used in the proof of Lemma 9, that
\[
\int_0^{\pi/2} J(x) \, dx = \int_0^{\pi/2} J^*(x) \, dx + (2n + 1)^{-1} \int_0^{\pi/2} [J(x) - J^*(x)] \, dx \\
- (2n + 1)^{-1} \int_0^{\pi/2} J^*(x) \, dx \\
= - (\pi/2) + (2n + 1)^{-1} T(-1)^n + (2n + 1)^{-1} \int_0^{\pi/2} [J(x) - J^*(x)] \, dx \\
+ O((2n + 1)^{-1}).
\]

An appropriate combination of these results shows that Lemma 10 is true, if we observe that \( f'(0) = 1/6, f'(\pi/2) = 4/\pi^2. \)

**Lemma 11.** Eq. (3.1) is true when \( r = 3 \) if

\[(3.33) \quad S_{30} = 0, \quad S_3 = 0.\]

We use the lemmas in section 2 to see that, for all positive \( z, \)

\[
G(z) v_3 = f(x) [T_2(z) + O(z^{-1})] + f(x) f'(x) [T_2'(z) + O(z^{-1})] \\
+ f''(x) [T_1(z) + O(z^{-1})] + f^3(x) [T_3(z) + O(z^{-1})] \\
+ f(x) \varphi(x) [T_1'(z) + O(z^{-1})] + \varphi'(x) O(z^{-1}).
\]

If \( F(x) \) is any one of the functions \( f(x), f(x)f'(x), \)
\( f''(x), f^3(x), f(x)\varphi(x), \) or \( \varphi'(x), \) so that \( F(x) = O(x) \) when \( 0 \leq x \leq \pi/2, \) then

\[
\varphi \int_0^{\pi/2} F(x) \sin qz \, dx = O((2n + 1)^{-1}),
\]

\[
\int_0^{\pi/2} F(x) O(z^{-1}) \, dx = O((2n + 1)^{-1}).
\]

Therefore, each of the terms in (3.34) contributes only \( O((2n + 1)^{-1}) \) to \( v_3. \) This remark completes the proof of Lemma 11.

The Theorem is now an immediate consequence of Lemmas 7, 8, 9, 10 and 11 if we define \( E_r \) so that
and observe that $S_0 + S_1 + S_2 + S_3 = 0$. If we use Lemma 2 and the identity
\[
\int_0^{\pi/2} \left( f^2(x) + 2x^{-1}f(x) \right) dx = \int_0^{\pi/2} (\csc^2 x - x^{-2}) dx = 2/\pi,
\]
we then find the following explicit formulas for $E_\ell$:
\[
E_0 = 1, \quad E_1 = 2/\pi \int_0^{\pi/2} (2^{1/2}G(z) - 1) dz,
\]
\[
E_2 = \left( \frac{1}{\pi^2} \right) - \left( \frac{5}{\pi} \right) \int_0^{\pi/2} x^{-1}f(x) dx + \left( \frac{2}{\pi} \right) \int_0^{\pi/2} \varphi(x) dx
- \frac{1}{2} - \left( \frac{5}{2\pi} \right) \int_0^{\pi/2} f^2(x) dx = -\frac{1}{2},
\]
\[
E_3 = \left( \frac{1}{3\pi} \right) \int_0^{\pi/2} \left( 2^{1/2}J(z) + 1 - 7 \cos z \right) dz
- \left( \frac{1}{3\pi} \right) \int_0^{\pi/2} \left( 2^{1/2}H(z) - 3 \sin z - (2z)^{-1}(5 + 11 \cos 2z) \right) dz.
\]
This completes the proof of the Theorem, except for the numerical values of $E_1$ and $E_3$. 

---

(3.35) \[ E_\ell = \sum_{m=0}^{\ell} S_{\ell-m,m} \quad (\ell = 0, 1, 2, 3), \]
CHAPTER 4
NUMERICAL RESULTS

We calculated the three integrals in (3.37) and (3.39) over the interval (0, 25π) by Simpson's rule. The integrals over the interval (25π, ∞) were calculated using the asymptotic relations (3.3), (3.9) and (3.10), (3.31) and (3.32), (3.25), (3.26) and (3.27) for G(z), H(z), J(z), K(z), M(z) and L(z), respectively. In each case it was desirable to include additional terms in those relations. The results for the coefficients E_1 and E_3 are those recorded in the statement of the Theorem, along with the values, E_0 = 1, E_2 = -1/2.

We have also numerically computed the integral in (2.5) when n = 2(1)60, using Simpson's rule. The results are displayed in Table 1. The approximation \( v'_n = 2^{-\frac{1}{4}}(2n + 1 + E_1) \) is in error (in excess) by about 0.0034% when \( n = 60 \), 0.0076% when \( n = 40 \), 0.030% when \( n = 20 \), 0.110% when \( n = 10 \), and 1.61% when \( n = 2 \). The more accurate approximation \( v''_n = 2^{-\frac{1}{4}}(2n + 1 + E_1 + (2n + 1)^{-1}E_2) \) is in error (by default) by about 0.00004% when \( n = 60 \), 0.00011% when \( n = 40 \), 0.00080% when \( n = 20 \), 0.00061% when \( n = 10 \), and 0.588% when \( n = 2 \). The most accurate approximation, \( v'''_n = 2^{-\frac{1}{4}}(2n + E_1 + (2n + 1)^{-1}E_2 + (2n + 1)^{-2}E_3) \), agrees to 5D with the correct
$v_n$ when $n \geq 8$, and is only in error (by default) by about 0.102\% when $n = 2$. These comparisons are displayed in Table 2.
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**TABLE 1.** 6D VALUES OF THE MEAN VALUE $v_n$ FOR $n = 1(1)60$. 
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**TABLE 2. TABLE OF APPROXIMATIONS $v_n'$, $v_n''$ AND $v_n'''$ TO $v_n$.**
WORKS CITED


