8-1-1963

Borel sets and baire functions

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BOREL SETS AND BAIRE FUNCTIONS

A THESIS
SUBMITTED TO THE FACULTY OF ATLANTA UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF MASTER OF SCIENCE
IN MATHEMATICS

BY
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ATLANTA, GEORGIA

AUGUST 1963
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CHAPTER I

INTRODUCTION

This thesis is an outgrowth of the course in Real Variables at Atlanta University. It can easily be understood if the reader has a fair knowledge of set theory and theory of real variables. The material has been strongly influenced by the work on set theory by Felix Hausdorff, the book on Real Functions by Casper Goofman and many lectures given by Dr. S. Saxena and Dr. L. Cross. Here you will be given some of the basic properties of open sets, closed sets and sets of $F_{\infty}$ and $G_{\infty}$, which are called Borel sets. Also, some properties of functions of type $f_{\infty}$, which are called Baire functions.

In view of the parallel way in which Borel sets and Baire functions have been defined and restricted to an ordinal number, the relationship between Borel sets and Baire functions is discussed.

Since all Borel sets are measurable sets, and Baire functions are measurable functions, therefore, some of the basic properties of measurable sets and measurable functions are discussed.
The ordinal number is the smallest infinite ordinal number. It has denumerable cardinal numbers. If we consider the set of ordinal numbers which have either finite or denumerable cardinals numbers, this set of ordinal numbers form a well-ordered set (i.e., every suborder has a first element). Thus, the well-ordered set $\mathcal{W}$ has an ordinal number which we designate by $\mathcal{A}$.

**Proposition.**—For every infinite ordinal number, the set of ordinal numbers less than is a well-ordered set whose ordinal number is $\mathcal{W}$.

**Proof.**—Let $\mathcal{W}^1$ be a well-ordered set whose ordinal number is $\alpha$. Let $\beta < \alpha$ and let $\mathcal{W}(\beta) = \mathcal{W}_b$ be the initial segment of $\mathcal{W}^1$ whose ordinal number is $\alpha$. This establishes a one-one order-preserving correspondence between the elements and its initial segments. (1:125)

Now $\mathcal{W}$ is the well-ordered set of all finite or denumerable ordinals $\mathcal{A}$. For every denumerable ordinal number $\alpha$, the above proposition shows that the initial segment $\mathcal{W}_\mathcal{A}$ of $\mathcal{W}$ has ordinal number $\alpha$. Hence $\mathcal{W} > \alpha$ for every denumerable ordinal number $\alpha$, so $\mathcal{W}$ is nondenumerable.
If $\alpha$ is a nondenumerable ordinal number and $W^1$ is a well-ordered set with ordinal number $\alpha$, then $W^1$ is not similar to any initial segment of $W$. Hence, $\alpha \geq \bigcap$. This shows that $\bigcap$ is the smallest nondenumerable ordinal number.

**Transfinite Induction.**—The method of transfinite induction for the positive integers is the following:

If a set $S$ of positive integers is such that $1 \in S$ and if $n \in S$ implies $n+1 \in S$, it follows that $S$ is the set of all positive integers.

**Theorem.**—If $S$ is a set of all ordinals less than a given ordinal $\alpha$, and $T$ is a subset of $S$ such that:

1. the first ordinal 1, is in $T$
2. for every ordinal $\beta < \alpha$, if all ordinals less than $\beta$ are in $T$ then $\beta$ is also in $T$, then $T = S$.

**Proof.**—If $S-T$ is nonempty, then there is a first ordinal $\beta \in S-T$ by (1) $\beta \neq 1$ but all ordinals less than $\beta \in T$. Hence, $S-T$ is empty. Therefore, $S-T = O$. Hence $S = T$. (1:131)

**Duality.**—If given any valid statement, and if we interchange the words open and closed sets, union and intersection, we get another valid statement.

If $A_\alpha$, $\alpha \in A$ is a system of sets, their intersection is the set consisting of those elements belonging to all the sets $A_\alpha$, $\alpha \in A$.

Thus $\chi \in A = \bigcap A_\alpha$ if and only if $\chi \in A_\alpha$ for every $\alpha \in A$. 
If \( A_1, A_2, A_3, \ldots, A_n, \ldots \) is an infinite sequence of sets (i.e., a set \( A_k \) associated with every positive integer \( k \)), their union \( \bigcup_{k=1}^{\infty} A_k \) is the set consisting of all the elements belonging to at least one of the sets \( A_k, k = 1, 2, 3, \ldots, n, \ldots \).

**Limits.**—If \( A_1, A_2, A_3, \ldots, A_n, \ldots \) is any sequence of sets, the set of all elements which belong to any infinite number of sets in the sequence is called its limit superior, and the set of all elements which belong to all but a finite number of sets in the sequence is called its limit inferior. The limit inferior is always a subset of the limit superior. If the limit inferior is equal to limit superior, the limit of the sequence of sets is said to exist. That is to say, the limit \( A_1, A_2, A_3, \ldots, A_n, \ldots \) exists if every element which belongs to an infinite number of sets belongs to all but a finite number of them.

**Convergent Sequence of Functions at a Point.**—The sequence \( \{f_n(x)\} \) of a function is said to converge at a point \( x_0 \) for \( \varepsilon > 0 \) there exist a positive \( N = N(\varepsilon, x_0) \)

\[ \exists \left| f_n(x_0) - f_m(x_0) \right| < \varepsilon \text{ for } n, m > N. \]

A sequence of function \( \{f_n(x)\} \) is convergent on a set if it is convergent at every point on the set, \( S \).

The sequence of function \( \{f_n(x)\} \) is uniformly convergent on a set \( S \), if for any \( \varepsilon > 0 \) there exist a positive \( N = N(\varepsilon) \)

\[ \exists \left| f_n(x) - f_m(x) \right| < \varepsilon \text{ for } m, n > N \quad \text{for every } x \in S. \]
Properties of Measurable Sets

1. If $S_1, S_2, \ldots, S_n$ is a sequence of set and
   
   $$ S = \bigcup_{n=1}^{\infty} S_n $$
   
   then
   
   $$ m_e(S) \leq \sum_{n=1}^{\infty} m_e(S_n) $$

2. If $G_1 \subseteq G_2 \subseteq \ldots \subseteq G_n \ldots$ is a nondecreasing sequence of open sets and $G = \bigcup_{n=1}^{\infty} G_n$, then for every $\varepsilon > 0$ there is an $n(\varepsilon) \geq m(G) < m(G) + \varepsilon$.

3. If $S_1, S_2, S_3, \ldots, S_n, \ldots$ is a denumerable number of measurable sets, then $S = \bigcup_{n=1}^{\infty} S_n$ is measurable.

4. If $S_1$ and $S_2$ are disjoint measurable sets and if $S = S_1 \cap S_2$ then $m(S) = m(S_1) + m(S_2)$.

5. If $S_1, S_2, \ldots, S_n, \ldots$ is a denumerable number of disjoint measurable sets and $S = \bigcup_{n=1}^{\infty} S_n$ then $m(S) = \sum_{n=1}^{\infty} m(S_n)$.

6. If $S_1, S_2, \ldots, S_n, \ldots$ is a denumerable number of measurable sets and $S = \bigcap_{n=1}^{\infty} S_n$, then $S$ is measurable.

7. If $S_1, S_2, \ldots, S_n, \ldots$ is a nonincreasing sequence of measurable sets and $S = \bigcap_{n=1}^{\infty} S_n$, then $m(S) = \lim_{n \to \infty} m(S_n)$.

Properties of Measurable Functions.--If $f(x)$ is measurable, if for every real number $k$ the following functions are measurable.
1. \(-f(x)\)
2. \(cf(x)\)
3. \(f(x) + c\)
4. \(\left[ f(x) \right]^2\)
5. \(\frac{1}{f(x)}\), if \(f(x) \neq 0\)
6. \(|f(x)|\)
7. \(f(x) + g(x)\)
8. \(f(x) \cdot g(x)\)
9. \(f(x) - g(x)\)
10. \(\frac{f(x)}{g(x)}\), if \(g(x) \neq 0\)

The proofs of the propositions are beyond the scope of this thesis. They can easily be found in most books on set theory.

Definitions

Definition 1. For every set \(S\), the number
\[ m_e(S) = \text{g.l.l.} \left[ m(G) \mid G \supset S \right] \]
where \(G\) varies over all open sets containing \(S\), is called the exterior measure of \(S\), where \(0 \leq m_e(S) \leq 1\).

Definition 2. For every set \(S\) the interior measure of \(S\) is the number
\[ m_i(S) = \text{l.u.l.} \left[ m(F) \mid F \subset S \right] \]
where \(F\) is a closed set.

Definition 3. If \(S\) is measurable, the number
\[ m(S) = m_e(S) = m_i(S) \]
is called the Lebesgue measure of \(S\).
CHAPTER III

BOREL SETS

The Borel sets are all the sets which can be obtained from the open and closed sets by repeatedly applying the operations of intersection and union to denumerable numbers of sets.

The closed sets are called of type $F_\infty$. Let $F_1$ be the union of a denumerable numbers of sets "of type $F_\infty$." Let $F_2$ be the intersection of a denumerable numbers of sets "of type $F_1$." Let $F_3$ be the union of a denumerable numbers of sets "of type $F_2$." Continuing in a similar manner, we can define a type of $F_\alpha$ for every $\alpha < \omega_1$.

Let $\alpha$ and $\beta$ be ordinal numbers $\omega$, where $\omega = \omega_1$ is the beginning number of the class $\mathcal{Z}(\mathcal{O})$. We again distinguish between even ordinals $2\alpha$, and odd ordinals $2\alpha + 1$. Hence, we have the following rule

\[
\alpha \left\{ \begin{array}{ll}
\text{The sets } F_\alpha \text{ are the sets in the systems } F_\beta & \text{if } \alpha \text{ is odd and the intersection if } \alpha \text{ is even, and } \\
\text{The sets } F_\alpha \text{ are the unions of sequence of sets } F_\beta & \text{if } \alpha > \beta. \quad (2:99)
\end{array} \right.
\]

Thus we see by induction we will obtain the sets $F_0$, $F_1$, $F_2$, ..., which can be written as $F_1$, $F_2$, $F_3$, $F_4$, ..., Hence we get $F_\alpha$ as intersections of sequence of preceding...
sets, the \( F_{\alpha+1} \) as unions of sequences of sets of \( F_{\alpha} \), and so forth.

A slightly different way of generating \( G \) is derived from the above by interchanging the roles played by union and intersection, so that the system \( G_\beta \) of sets \( G \) are defined as follows:

\[
\begin{align*}
\{ & \text{The set } G \text{ are the open sets in the system } \\
& \text{The set } G_\alpha \text{ are the intersection if } \alpha \text{ is odd and} \\
& \text{the union if } \alpha \text{ is even, of sequences of sets } \\
& G_\beta, \text{ where } \alpha > \beta.
\end{align*}
\]

All the sets \( G_\alpha \) also form the smallest Borel system \( G \). The \( G_{\alpha+1} \) are the intersection or union (depending on whether \( \alpha \) is even or odd) of sequence of set \( G_\alpha \). Hence, by induction, we obtain the sets \( G_0, G_1, G_2, \ldots \) which can be written as \( G_1, G_5, G_6, \ldots \).

Proposition 1: The complement of every set of type \( F_\alpha \) is of type \( G_\alpha \) and the complement of every set of type \( G_\alpha \) is of type \( F_\alpha \) for every \( \alpha \in \mathbb{N} \).

Proof: The proposition holds for \( \alpha = 0 \). Suppose it holds for every \( \beta < \alpha \) and suppose \( \alpha \) is even. Let \( S \) be any set of type \( F_\alpha \), then \( S = \bigcap_{n=1}^{\alpha} S_n \), where \( S_n \) is of type \( F_{\alpha_n} \), \( \alpha_n < \alpha \) by hypothesis. \( c(S_n) \) is of type \( G_{\alpha_n} \). But \( c(S) = \bigcup_{n=1}^{\alpha} c(S_n) \), so that \( c(S) \) is of type \( G_\alpha \). Hence \( c(F_\alpha) \) is of type \( G_\alpha \). Q.E.D. (1:135)

Part II: The complement of every set of \( G_\alpha \) is of type \( F_\alpha \). Suppose \( \alpha \) is even. Let \( S \) be any set of type \( G_\alpha \).
then \( S = \bigcup_{n=1}^{\infty} S_n \), where \( S_n \) is of type \( G^{\alpha_n} \), \( \alpha_n < \alpha \). But by hypothesis \( c(S_n) \) is of type \( F^{\alpha_n} \). But \( c(S) = \bigcap S_n \), so \( c(S) \) is of type \( G^\alpha \). Hence \( c(S) \) is of type \( F^\alpha \). Q.E.D.

The cases where \( \alpha \) is odd can be proved by the principle of duality.

**Proposition 2:** If \( \alpha < \omega \) is odd, the union of a denumerable numbers of sets of \( F^\alpha \) is of type \( F^\alpha \) and the intersection of a denumerable number of type \( G^\alpha \) is of type \( G^\alpha \).

**Proof:** Let \( S = \bigcup_{n=1}^{\infty} S_n \) where each \( S_n \) is of type \( F^\alpha \), then \( S_n = \bigcup_{m} S_{nm} \) where each \( S_{nm} \) is of type \( F^{\alpha_{nm}} \). Hence \( S = \bigcup_{n \in \omega} \bigcup_{m} S_{nm} \), as the union of a denumerable number of sets of type \( F^{\alpha_{nm}}, \alpha_{nm} < \alpha \) is of type \( F^\alpha \). Therefore \( F^\alpha \) is of type \( F^\alpha \). Q.E.D. (1:135)

Part II: The intersection of a denumerable numbers of sets of type \( G^\alpha \) is of type \( G^\alpha \).

**Proof:** Let \( \alpha \) be an odd ordinal number and let \( S = \bigcap_{n=1}^{\infty} S_n \) where each \( S_n \) is of type \( G^\alpha \), then \( S_n = \bigcap_{m} S_{nm} \) where each \( S_{nm} \) is of type \( G^{\alpha_{nm}} \). \( \alpha_{nm} < \alpha \). Hence \( S = \bigcap_{n \in \omega} \bigcap_{m} S_{nm} \), as the intersection of a denumerable number of sets of type \( G^{\alpha_{nm}}, \alpha_{nm} < \alpha \) is of type \( G^\alpha \). Therefore \( G^\alpha \) is of type \( G^\alpha \). Q.E.D.

The proof for even ordinal numbers can be proved by the principle of duality.

**Proposition 3:** For every \( \alpha < \omega \), the union and intersection of any finite number of sets of type \( F^\alpha \) (of type \( G^\alpha \))
is of type $\mathcal{F}_\alpha$ (of type $\mathcal{G}_\alpha$).

**Proof:** Let $\alpha$ be an even ordinal number and let $S_n$ be the sets of type $\mathcal{F}_\alpha$, then $S = \bigcap_{m=1}^{\alpha} \bigcap_{n=1}^{\alpha} S_{nm}$ where $S_{nm}$ are sets of type $\mathcal{F}_{\alpha \cdot n}$, $\alpha_{nm} \prec \alpha$, then $S$ is the intersection of a denumerable number of sets of type $\mathcal{F}_{\alpha n}$, $(\alpha_n \prec \alpha)$ is of type $\mathcal{F}_\alpha$. Therefore the intersection of a finite number of sets of the type $\mathcal{F}_\alpha$ are of type $\mathcal{F}_\alpha$. Q.E.D.

**Part II:** Let $\alpha$ be an even ordinal number and let $S$ be sets of type $\mathcal{G}_\alpha$, then $S = \bigcup_{m=1}^{\alpha} \bigcup_{n=1}^{\alpha} S_{nm}$ where each $S_{nm}$ is of type $\mathcal{G}_{\alpha \cdot n}$, $\alpha_{nm} < \alpha$, then $S$ is the union of a denumerable number of sets of type $\mathcal{G}_{\alpha n}$, $(\alpha_n < \alpha)$ is of type $\mathcal{G}_\alpha$. Therefore a finite number of sets of type $\mathcal{G}_\alpha$. Therefore a finite number of sets of type $\mathcal{G}_\alpha$ are of type $\mathcal{G}_\alpha$. Q.E.D.

**Proposition 4:** For every $\alpha$, every set of type $\mathcal{F}_\alpha$ is of type $\mathcal{G}_\alpha + 1$ and every set of type $\mathcal{G}_\alpha$ is of type $\mathcal{F}_\alpha + 1$.

**Proof:** The proposition holds for $\beta < \alpha$. Given (Every $\mathcal{F}_\beta$ type is a $\mathcal{G}_\alpha$ type and every $\mathcal{G}_\alpha$ type is an $\mathcal{F}_\alpha$ type). Note: Every closed set is the intersection of a sequence of open sets; every open set is the union of a sequence of closed sets. Since $\mathcal{G}_\alpha$ is of type $\mathcal{F}_1$ let $S$ be of type $\mathcal{G}_1$, then $S$ is the union of a finite or denumerable number of open sets. But every open set is the union of a finite or denumerable number of closed sets. Hence $S$ is
the union of a finite or denumerable number of closed sets so $S$ is of type $F_1$. Let $\alpha \in \mathbb{N}$ and $\beta < \alpha$ then every set of $G_\beta$ is of type $F_{\beta + 1}$. Let $S$ be of type $G_\alpha$ then $S = \bigcap_{n=1}^\infty S_n$, where each $S_n$ is of type $G_{\alpha n}$, $\alpha n < \alpha$. By hypothesis $S_n$ is of type $F_{\alpha n + 1}$ with $\alpha n + 1 < \alpha + 1$, since $1 + \alpha$ is even $S = \bigcap_{n=1}^\infty S_n$ is of type $F_{\alpha + 1}$. Hence $G_\alpha$ is of type $F_{\alpha + 1}$. Q.E.D. (1:136)

Part II: Every set of type $F_\alpha$ is of type $G_{\alpha + 1}$.

Proof: Since every set of type $G_1$ is of type $F_0$. Let $S$ be sets of type $F_1$ then $S$ is closed. Hence $S$ is the intersection of a finite or denumerable number of closed sets. But every closed set is the intersection of a denumerable number of open sets. Hence $S$ is the intersection of a finite or denumerable number of open sets, so $S$ is of type $G_1$. Let $\alpha \in \mathbb{N}$, $\beta < \alpha$, then every set of $F_\beta$ is of type $G_{\beta + 1}$. Let $S$ be of type $F_\alpha$ then $S = \bigcup_{n=1}^\infty S_n$, where each $S_n$ is of type $F_{\alpha n}$, $\alpha n < \alpha$. By hypothesis $S_n$ is of type $G_{\alpha n + 1}$ with $\alpha n + 1 < \alpha + 1$ since $\alpha + 1$ is even, $S = \bigcup_{n=1}^\infty S_n$ is of type $G_{\alpha n + 1}$. Hence $F_\alpha$ is of type $G_{\alpha + 1}$. Q.E.D.

Proposition 5: The Borel sets form the smallest system of set such that

1. all closed sets are in the system,
2. the union of any denumerable number of sets in the system is in the system,
3. The intersection of any denumerable number of sets in the system is in the system.
Proof: Let $S$ be any system which satisfies (1), (2), (3). Then every set of type $F_\alpha$ is in $S$. The method of transfinite induction and properties (2) and (3) assure that, for every $\alpha \in \Omega$, every set of type $F_\alpha$ is in $S$. (1:137)

Proposition 6: All Borel sets are measurable.

Proof: The closed and open sets are measurable. Let $\beta < \alpha$ and suppose every $\beta < \alpha$, every set of type $F_\beta$ and $G_\beta$ is measurable. Since the union of a denumerable number of measurable sets is measurable, then $S = \bigcup_{n=1}^{\infty} S_n$ is measurable. Also every set of type $F_\alpha$ and of type $G_\alpha$ is measurable, since the intersection of a denumerable number of measurable sets $S = \bigcap_{n=1}^{\infty} S_n$, then $S$ is measurable.
CHAPTER IV

BAIRE FUNCTIONS

The Baire Functions are a class of functions similar to the Borel sets. The continuous functions are said to be of type $f_0$ (Baire Class 0). Functions which are limits of convergent sequence of continuous functions are of type $f_1$ (Baire Class 1), for every $\alpha < \Omega$. If the functions are of type $f$ have been defined for every $\beta < \alpha$, then the functions of type $f_\alpha$ (Baire Class $\alpha$) are limits of convergent sequences of functions of type $\beta < \alpha$.

Baire has introduced an important classification of functions as follow:

Let $f(x)$ be defined over $U$; $f$ and $U$ are bounded or unbounded. If $f$ is continuous in $U$, we say the class is 0 in $U$ and write class $f \equiv 0$, mod $U$, then $f(x) = \lim_{n \to \infty} f_n(x)$ where each $f_n$ being of class 0 in $U$. We say its class is 1 if $f$ does not lie in the class 0 mod.

Let the series $F(x) = \sum f_n(x)$ converge in $U$, each term $f_n$ being continuous in $U$. Since $F(x) = \lim_{n \to \infty} F_n(x)$ we see $F$ is of class 0, or 1, according as $F$ is continuous or not continuous in $U$. A similar remark holds for infinite products:

$$G(x) = \prod g_n(x)$$
The derivatives of a function $f(x)$ give rise to functions of class $0$ or $1$. Let $f(x)$ have a unique differential coefficient $g(x)$ at each point of $U$. Both $f$ and $U$ may be unbounded. To fix the ideas, suppose the right-hand differential coefficient exists. Let $h_1 > h_2 > \ldots = 0$. Then

$$g_n(x) = \frac{f(x + h_n) - f(x)}{h_n}, \quad x + h_n \in U$$

is a continuous function of $x$ in $U$, but $g(x) = \lim_{n \to \infty} g_n(x)$ exist at each $x$ in $U$ by hypothesis.

**Proposition 1:** For every $\alpha < \alpha$, the sum and product of two functions of type $f_\alpha$ is of type $f_\alpha$.

**Proof:** The proposition holds for $\alpha = 0$. Suppose and the proposition holds for all $\beta < \alpha$. Let $f(x)$ and $g(x)$ be the two functions of type $f_\alpha$. Then $f(x) = \lim_{n \to \infty} f_n(x)$ and $g(x) = \lim_{n \to \infty} g_n(x)$ where, for every $n$, $f_n(x)$ is of type $f_{\alpha_n}$, $\alpha_n < \alpha$, and $g_n(x)$ is of type $\beta < \alpha$, so that $f_n(x) + g_n(x)$ and $f_n(x) \cdot g_n(x)$ are of type $\max(\alpha_n, \beta_n)$. Since $f(x) + g(x) = \lim_{n \to \infty} [f_n(x) + g_n(x)]$ and $f_n(x) \cdot g_n(x) = \lim_{n \to \infty} (f_n(x) \cdot g_n(x))$, it follows then that $f(x) + g(x)$ and $f(x) \cdot g(x)$ are of type $f_\alpha$. Q.E.D. (1:137)

**Proposition 2:** If for every $\alpha < \Omega$, $f(x)$ is of type $f_\alpha$ and $f(x) \neq 0$, then $\frac{1}{f(x)}$ is of type $f_\alpha$ and $-f(x)$ is of type $f_\alpha$.

**Proof:** For every $\alpha < \Omega$, $f(x) \neq 0$ and the proposition holds for every $\beta < \alpha$. Let $\frac{1}{f(x)}$ be of type $f_\alpha$, then $\frac{1}{f(x)} = \lim_{n \to \infty} \frac{1}{f(x)}$, where each $\frac{1}{f_n(x)}$ is of type $\frac{1}{f_\alpha}$ where $\alpha_n = 1$. Let $\alpha_n = 1$. 
then each $\frac{1}{f_n(x)}$ is of type $f_{\alpha n}$. Now $f(x) = \lim_{n \to \infty} f_n(x)$ such that $\frac{1}{f(x)}$ is of type $f_{\alpha}$. Q.E.D.

**Part II:** If $f(x)$ is of type $f_{\alpha}$ for every $\alpha < \Omega$ and the proposition holds for every $\beta < \alpha$, then let $-f(x)$ be of type $f_{\alpha}$. Hence, $-f(x) = \lim_{n \to \infty} -f_n(x)$, where each $-f_n(x)$ is of type $-f_{\alpha n}, \alpha_n < \beta$. Hence $(-1) \left[ -f(x) = \lim_{n \to \infty} -f_n(x) \right] = \left[ f(x) = \lim_{n \to \infty} f_n(x) \right]$. Therefore $-f(x)$ is of type $f_{\alpha}$. Q.E.D.

**Proposition 3:** For every $\alpha < \Omega$ if $f(x)$ is of type $f_{\alpha}$, then $|f(x)|$ is of type $f_{\alpha}$.

**Proof:** The proposition holds for functions of type $f_0$. Suppose $\alpha < \Omega$. It holds for all functions of type $\beta < \alpha$. Let $|f(x)|$ be of type $f_{\alpha}$. Then $|f(x)| = \lim_{n \to \infty} |f_n(x)|$, where each $f_n(x)$ is of type $f_{\alpha n}, \alpha_n < \beta$, but then each function $|f_n(x)|$ is of type $f_{\alpha n}$. Each $|f(x)| = \lim_{n \to \infty} |f_n(x)|$, so then $f(x)$ is of type $f_{\alpha}$ (by transfinite induction). Q.E.D.

**Proposition 4:** For every $\alpha < \Omega$, if $f(x)$ and $g(x)$ are of type $f_{\alpha + 1}$, then max $(f(x), g(x))$ and min $(f(x), g(x))$ are of type $f_{\alpha}$. 

**Proof:** The functions $f(x) + g(x)$ and $|f(x) - g(x)|$ are of type $f_{\alpha}$. Hence the functions $\max f(x), g(x) = \frac{1}{2} \left( f(x) + g(x) \right) + \frac{1}{2} |f(x) - g(x)|$ and $\min f(x), g(x) = \frac{1}{2} \left( f(x) + g(x) \right) - \frac{1}{2} |f(x) - g(x)|$ are of type $f_{\alpha}$. Q.E.D. (1:138)
Theorem 1: For every $\alpha < \Omega$, the limit of a uniformly convergent sequence of functions of type $f_\alpha$ is of type $f_\alpha$.

The proof of this theorem depends on the following lemma.

Lemma: For every $\alpha < \Omega$, if $f(x)$ is of type $f_\alpha$ and $|f(x)| \leq k$ for every $x$ where $k > 0$, then $f(x) = \lim_{n \to \infty} f_n(x)$, where each $f_n(x)$ is of type $f_{\alpha_n}, \alpha_n < \alpha$, and $|f_n(x)| \leq k$ for every $x$.

Proof: Since $f(x)$ is of type $f_\alpha$, there is a sequence $\{g_n(x)\}$ the limit of $g_n(x) = f(x)$, where $g_n(x)$ is of type $f_{\alpha_n}, \alpha_n < \alpha$. Let $h_n(x) = \min \{g_n(x), k\}$ and $f_n(x) = \max \{h_n(x), -k\}$. By the above Proposition, $f_n(x)$ is of type $f_{\alpha_n}$; therefore, $f(x) = \lim_{n \to \infty} f_n(x)$.

Proof of Theorem: If the proposition holds for $\alpha = 0$, suppose it holds for every $\beta < \alpha$. Let $\{f_n(x)\}$ be a uniformly convergent sequence of functions of type $f_\alpha$ and let $f(x) = \lim_{n \to \infty} f_n(x)$.

Now
\[ f(x) = \sum_{n=0}^{\infty} f_{n+1}(x) - f_n(x) + f_0(x), \]

where the series converges uniformly to $f(x)$. Hence there is a convergent series $\sum_{n=0}^{\infty} k_n$ of positive numbers such that for every $n$ and $x$, $|f_{n+1}(x) - f_n(x)| \leq k$.

For every $n$ the function $f_{n+1}(x) - f_n(x)$ if of type $f_{\alpha_n}$. Hence there is a sequence $\{f_{nm}(x)\}$ of functions of lower type $f_{\alpha_n}$, which converges to $f_{n+1}(x) - f_n(x)$, such
that for every \( m \) and \( x \) \( \left| f_{nm}(x) \right| \leq k_n \). Consider the sequence \( \left\{ g_n(x) \right\} = f_{11}(x), g_2(x) = f_{22}(x), \ldots, g_m(x) = f_{mm}(x) \)...

...for every \( n \), where \( g_n(x) \) is of lower type than \( f_\alpha \). We shall show that \( f(x) = \lim_{n \to \infty} g_m(x) \). Let \( \varepsilon > 0 \) a positive \( N \) such that \( \sum_{n=N+1}^{N} k_n < \varepsilon / 3 \). Hence for every \( x \), \( \left| f(x) - f_1(x) - \sum_{n=N+1}^{N} f_{n+1}(x) - f_n(x) \right| < \varepsilon / 3 \), \( n > N \) and \( m > N', \left| f_{n+1}(x) - f_n(x) - f_{nm}(x) \right| < \varepsilon / 3 \). Let \( m > \max(N,N') \). Then \( \left| f(x) - g_m(x) \right| < \left| f(x) - f_1(x) - \sum_{n=N+1}^{N} f_{n+1}(x) - f_n(x) \right| < \varepsilon / 3 \). \( \varepsilon / 3 \) **Q.E.D.** (1: 139)

**Theorem 2:** For any \( \alpha \in \Omega \), if \( f(x) \) is of class \( f_\alpha \), there is a set \( S \) whose complement is of the first category, such that \( f(x) \) is continuous on \( S \) relative to \( S \).

**Proof:** Let \( \alpha \in \Omega \) and suppose it holds for every \( \beta \prec \alpha \). Let \( f(x) \) be of type \( \alpha_n \prec \alpha \). Every \( n \) is of type \( S_n \) whose complement is of the first category such that \( f_n(x) \) is continuous on \( S_n \) relative to \( S_n \). But the complement of \( S = \bigcap_{n=1}^{\infty} S_n \) is of the first category and \( f_n(x) \) is continuous on \( S \) relative to \( S \) for every \( n \). \( \varepsilon / 3 \) **Q.E.D.**

**Theorem 3:** The Baire functions are \( c \) in number.

**Proof:** There are \( c \) continuous functions because a continuous function is completely determined by its values at the rationals. Hence there are \( c^c = \left(2^\aleph_0 \right)^{\aleph_0} = c \).
continuous functions. So the functions of type $f_0$ are $c$ in number. Suppose $\alpha<\Omega$ and for every $\beta<\Omega$ there are $c$ or fewer functions of type $f_\beta$. The class of all functions whose type is less than has cardinal number $c^\mathcal{N}_0 = c$. But every function of type $f_\alpha$ is the limit of a sequence of function of lower type so that the number of function of type $f_\alpha$ is no more than $c^\mathcal{N}_0 = c$. By induction this holds for every $\alpha<\Omega$ hence there is no more than $c^\mathcal{N}_0 = c$ Baire functions. But there are at least $c$ Baire functions, since the functions $f(x) = a$ are Baire functions.

**Theorem 4:** Every Baire function is measurable.

**Proof:** If $f(x)$ is a Baire function, then for every real number $k$ the set $S = E \left\{ f(x) > k \right\}$ is a Borel set and so is measurable. Q.E.D.
CHAPTER V

RELATION BETWEEN BOREL SETS AND BAIRES FUNCTIONS

Definition 1. \( f(x) \) is said to be continuous on \( S \) if it is continuous at every point of \( S \).

Theorem 1. (a): \( f(x) \) is continuous on \( U \) if and only if for every \( k \), the sets \( E[f(x) > k] \) and \( E[f(x) < k] \) are open sets.

Proof: Suppose \( f(x) \) is continuous on \( U \). Let \( \xi \in U \) and \( f(\xi) > k \) there is an \( \epsilon > 0 \) \( \Rightarrow \) \( |f(x) - f(\xi)| < \epsilon \). Since \( f(x) \) is continuous at \( \xi \), there is a \( \delta > 0 \) if \( |x - \xi| < \delta \) whenever \( |f(x) - f(\xi)| < \epsilon \). Hence there is a neighborhood \( I \) of \( \xi \) for every \( \xi \in I \), \( f(x) > f(\xi) - \epsilon > k \). Thus, \( E[f(x) > k] \) and \( E[f(x) < k] \) are open sets.

Suppose the sets \( E[f(x) > k] \) and \( E[f(x) < k] \) are open for every \( k \). Let \( \xi \in U \) and let \( \epsilon > 0 \). Then, since \( E[f(x) > f(\xi) - \epsilon] \) is open, There is a neighborhood \( I \) of \( \xi \) such that for every \( \xi \in I \), \( f(x) > f(\xi) - \epsilon \). Moreover, since \( E[f(x) > f(\xi) + \epsilon] \) is open. There is a neighborhood of \( \xi \) such that for every \( \xi \in I_2 \), \( f(x) > f(\xi) + \epsilon \). \( I = I_1 \cap I_2 \) is a neighborhood of \( \xi \). For every \( \xi \in I \), \( |f(x) - f(\xi)| < \epsilon \) so that \( f(x) \) is continuous at \( \xi \).
Theorem 1 (b): \( f(x) \) is continuous on \( U \) if and only if for every \( k \), the sets \( E\{ f(x) \geq k \} \) and \( E\{ f(x) \leq k \} \) are closed.

Proof: Since the complement of an open set is a closed set. If \( f(x) \) is continuous then sets \( E\{ f(x) > k \} \) and \( E\{ f(x) < k \} \) are open sets (Theorem 1 (a)). The complements of two sets \( E\{ f(x) > k \} \) and \( E\{ f(x) < k \} \) are the sets \( E\{ f(x) \leq k \} \) and \( E\{ f(x) \geq k \} \) respectively. Therefore, the sets \( E\{ f(x) > k \} \) and \( E\{ f(x) < k \} \) are closed sets. Conversely, if \( E\{ f(x) \leq k \} \) and \( E\{ f(x) \geq k \} \) are closed sets, then sets \( E\{ f(x) < k \} \) and \( E\{ f(x) > k \} \) are open, therefore \( f(x) \) is continuous. Q.E.D.

There is a connection existing between Borel sets and Baire functions because of the parallel way in which they have been constructed as well as defined. The sets associated with functions of finite Baire type are of finite Borel type and conversely that if sets associated with a function are all of the same finite Borel type then the function is of finite Baire type. That is to say that a function \( f(x) \) is of type \( f_\alpha \) iff for every real number \( k \) the sets \( E\{ f(x) > k \} \) and \( E\{ f(x) < k \} \) are of type \( G_\alpha \) and \( F_\alpha \) respectively, if \( \alpha \) is an even positive integer and of type \( F_\alpha \) and \( G_\alpha \) respectively, if \( \alpha \) is an odd positive integer.

For convenience let sets of type \( A_\alpha \) and \( B_\alpha \) are defined. For every \( \alpha \in \Omega \), a set \( S \) will be of type \( A_\alpha \) if there is a function \( f(x) \) of type \( f_\alpha \) and a real number \( k \) such that
$S = E \left[ f(x) > k \right]$ and a set, $S$ will be of type $B_{\alpha}$ if there is a function $f(x)$ of type $f_{\alpha}$ and a real positive number $k$ such that $S = E \left[ f(x) \geq k \right]$.

**Lemma 2:** If $f(x) = \lim f_n(x)$ then

$$E \left[ f(x) > k \right] = \bigcup_{n=1}^{\infty} \bigcap_{r=1}^{n} E \left[ f_n(x) \geq k + \frac{1}{m} \right].$$

**Proof:** Suppose $f(x) > k$. Then $\exists m \geq f(x) > k + \frac{1}{m}$. Hence if $x_0 \in E \left[ f(x) > k \right]$ then

$$x_0 \in \bigcup_{n=1}^{\infty} \bigcap_{r=1}^{n} E \left[ f_n(x_0) \geq k + \frac{1}{m} \right].$$

Suppose that

$$x_0 \in \bigcup_{n=1}^{\infty} \bigcap_{r=1}^{n} E \left[ f_n(x_0) \geq k + \frac{1}{m} \right].$$

Then there is an $m$

$$x_0 \in \bigcup_{n=1}^{\infty} \bigcap_{r=1}^{n} E \left[ f_n(x_0) \geq k + \frac{1}{m} \right].$$

There is also an $r$

$$x_0 \in \bigcap_{n=r}^{\infty} E \left[ f_n(x_0) \geq k + \frac{1}{m} \right].$$

Thus

$$x_0 \in E \left[ f_n(x_0) \geq k + \frac{1}{m} \right] \subseteq E \left[ f(x) > k \right].$$

This shows that

$$E \left[ f_n(x) \geq k + \frac{1}{m} \right] = E \left[ f(x) > k \right].$$

(1.141)

**Proposition 1:** For every finite ordinal $\alpha$, every set of type $A_{\alpha}$ is of type $G_{\alpha}$ and every set of type $B_{\alpha}$ is of type $F_{\alpha}$ if $\alpha$ is even. Every set of type $A_{\alpha}$ is of type $E_{\alpha}$ and every set of type $B_{\alpha}$ is of type $G_{\alpha}$ if $\alpha$ is odd.
Proof: The proposition holds for $\alpha = 0$. Suppose it holds for every $\beta < \alpha$. Suppose $\alpha$ is odd. Let $S$ be a set of type $A_\alpha$. There is then a function $f(x)$ of type $f_\alpha$ and a positive real number $k \ni S = \bigcap f(x) > k$. Now $f(x) = \lim_{n \to \infty} f_n(x)$ where the $f_n(x)$ are Baire functions of lower type than $f_\alpha$. By Lemma 2,

$$S = \bigcup_{n=1}^\infty \bigcup_{m=1}^\infty \bigcap \big( f_n(x) \geq k + \frac{1}{m} \big)$$

But, by assumption, each $\bigcap f_n(x) \geq k + \frac{1}{m}$ is of type $F_\alpha - 1$. Since $\alpha - 1$ is even, the intersection of a denumerable number of sets of type $F_\alpha - 1$ is of type $F_\alpha - 1$. Hence $S$ is the union of a denumerable number of sets of type $F_\alpha$ is of type $F_\alpha$. (1:141)

Suppose $S$ is of type $B_\alpha$. There is a function $f(x)$ of type $f_\alpha$ and a real number $k \ni S = \bigcap f(x) > k$. Hence $S = \bigcap f(x) > k$. But $S = \bigcap [\neg f(x) \leq k]$. $c(S) = \bigcap [\neg f(x) \geq k]$, so that $c(S)$ is of type $F_\alpha$, whence $S$ is of type $G_\alpha$. If $\alpha$ is a finite even ordinal the proof is similar.

Lemma 3: For every finite ordinal $\alpha$, if $S$ is the set of type $A_\alpha$ or $B_\alpha$, there is a function $f(x)$ of type $f_\alpha + 1$, $f(x) = 1$ for every $x \in S$ and $f(x) = 0$ for every $x \notin S$.

Proof: Suppose $S$ is of type $A_\alpha$. Then there is a function $g(x)$ of type $f_\alpha$. $S = \bigcap g(x) > 0$. Let $h(x) = \max \big[ g(x), 0 \big]$. Then $h(x)$ is also of type $f_\alpha$. For every positive integer $n$ let $f_n(x) = \min \big[ nh(x), 1 \big]$. The function $f_n(x)$ are of type $f_\alpha$. The sequence $f_n(x)$ converges
everywhere and \( \lim f_n(x) = 1 \) for \( x \in S \) and 0 for \( x \notin S \) and is of type \( f_{\alpha+1} \).

Suppose \( S \) is of type \( B_\alpha \). Then \( c(S) \) is of type \( A_\alpha \). Hence there is a function \( f(x) \) of type \( f_{\alpha+1} \) \( f(x) = 1 \) for \( x \in S \) and \( f(x) = 0 \) for \( x \notin S \). The function \( 1 - f(x) \) is of type \( f_\alpha + 1 \) and is 1 on \( S \) and 0 on \( c(S) \). (1:142)

**Proposition 2:** For every finite ordinal \( \alpha \), every set of type \( G_\alpha \) is of type \( A_\alpha \) and every set of type \( F_\alpha \) is of type \( B_\alpha \) if \( \alpha \) is even. Every set of type \( F_\alpha \) is of type \( A_\alpha \) and every set of type \( G_\alpha \) is of type \( B_\alpha \) if \( \alpha \) is odd.

**Proof:** The proposition holds for \( \alpha = 0 \). Suppose \( \alpha \) is odd and the proposition holds for \( \alpha - 1 \). Let \( S \) be the set of type \( F_\alpha \). Then \( S = \bigcup S_n \) where each \( S_n \) is of type \( F_\alpha \) and the sequence \( \{S_n\} \) is nondecreasing. Since by assumption every set of type \( F_{\alpha-1} \) is of type \( B_{\alpha-1} \), by Lemma 3, a function \( f_n(x) \) of type \( f_\alpha \) for every \( n \) \( f_n(x) = \frac{1}{2^n} \) for every \( x \in S_n \) and \( f_n(x) = 0 \) for every \( x \notin S_n \). Hence,

\[
\int_0 1(x) = \sum_{n=1}^{\infty} f_n(x)
\]

is of type \( f_\alpha \) since the series converges uniformly. But \( S = E \{f(x) > 0\} \), so \( S \) is of type \( A_\alpha \). Similarly, every set of type \( G_\alpha \) is of type \( B_\alpha \). (1:142)

**Part II:** If \( \alpha \) is even and the proposition holds for \( \alpha - 1 \), want to show that every set of type \( F_\alpha \) is of type \( B_\alpha \) and every set of \( G_\alpha \) is of type \( A_\alpha \).

**Proof:** The proposition holds for \( \alpha - 1 \) and suppose it holds for \( \alpha = 0 \). Let \( S \) be set of type \( G_\alpha \). Then \( S = \bigcup \)
where each $S_n$ is of type $G_\alpha$ and the sequence $\{S_n\}$ is non-decreasing. Since every set of type $G_\alpha$ is of type $A_\alpha$, by Lemma 3, a function $f_n(x)$ of type $f$ for every $n$ \( f_n(x) = \frac{1}{2^n} \) for every $x \in S_n$ and $f_n(x) = 0$ for every $x \not\in S$. Hence

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

is of type $f_\alpha$, since the series converges uniformly. But $S = E[f(x) > 0]$ so $S$ is of type $A_\alpha$. Hence set of $G_\alpha$ is of type $A_\alpha$. Similarly, every set of $F_\alpha$ is of type $B_\alpha$.

Q.E.D.

From the previous proven propositions, the following theorems are obtained:

**Theorem 4**: For finite odd ordinals, a set is of type $A_\alpha$ iff it is of $E_\alpha$, and is of type $B_\alpha$ iff it is of $G_\alpha$. For finite even ordinals a set is of type $A_\alpha$ iff it is of type $G_\alpha$ and is of type $B_\alpha$ iff it is of type $F_\alpha$.

**Theorem 5**: If $f(x)$ is any function of type $f_\alpha$, where $\alpha$ is a finite ordinal, then for every real number $k$ the set $E[f(x) > k]$ and $E[f(x) \geq k]$ are of type $F_\alpha$ and $G_\alpha$ respectively if $\alpha$ is odd and of type $G_\alpha$ and $F_\alpha$ respectively if $\alpha$ is even.

This is proved by Theorem 4. (1:143)
APPENDIX

SYMBOLS

\[ \Omega \]
iff
\[ \geq \]
\[ \leq \]
\[ \not= \]
\[ (1:25) \]
\[ \in \]
\[ \exists \]
\[ \epsilon \]
\[ \aleph \]
union
\[ \delta \]
intersection
\[ \alpha \]
\[ \{f_n(x)\} \]
sequence of functions
\[ \beta \]
\[ \lim \]
\[ \sum \]
\[ \infty \]

Omega
if and only if
greater than or equal to
less than or equal to
not equal
reference
contained in
such that
there exist
epsilon
Aleph zero
union
delta
intersection
alpha
sequence of functions
beta
limit
summation
infinity
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Unpublished Materials

Lectures given in class by D. S. Saxena and Dr. L. Cross.