Vector methods in algebra and geometry

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VECTOR METHODS IN ALGEBRA AND GEOMETRY

A THESIS
SUBMITTED TO THE FACULTY OF ATLANTA UNIVERSITY
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FOR THE DEGREE OF MASTER OF ARTS

BY

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DEPARTMENT OF MATHEMATICS

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TO

MRS. RUTH DARNARD
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CHAPTER I
INTRODUCTION

The word vector is the Latin word for carrier; and was defined in this way in 1846 by the eminent Scottish Mathematician, Sir William Rowan Hamilton, who desired to convey the idea that he was looking upon a vector as an operator which carries a particle from the initial point to the terminal point.

There had been a growing feeling that the processes of analysis were in some way artificial and complex. It is chiefly through the labors of Gibbs and Heaviside that analysis has been perfected which not only does away with the complexity and artificiality of other analysis but offers a strictly natural and therefore as direct and simple a substitute as possible, and, at the same time in no wise is at variance, but runs parallel, to them. This new, yet old method is Vector Analysis.

Vector Analysis may be considered under three divisions:

(1) Vector Algebra
(2) Vector Geometry
(3) Vector Calculus

The writer makes use of the elementary operations and attempts to relate and illustrate its power and usage pertaining to some theorems and formulas which may be proved or verified by means of Vector Algebra and Vector Geometry. Stress is placed on logical aspects more than on technique and details of calculation. The laws governing operations with vectors are then postulated and discussed.

The notations adopted are that of Professor Willard Gibbs, one of the great American physicists and mathematicians.
CHAPTER II

FUNDAMENTAL DEFINITIONS

In the study of Mathematical courses that preceded the study in Vector Analysis, when numbers, points, and lengths on a line are used to represent physical and geometrical concepts, it is unimportant what line is used; that is, the line may be drawn in any convenient direction. We spoke in terms of numbers only or numbers and an algebraic sign. We now speak of the representation of certain physical concepts by line segment where the direction of the line segment is of much importance.

1. A Vector is a directed segment of a straight line on which are distinguished an initial and a terminal point. A Vector thus has a magnitude and a direction. Any quantity which can be represented by such a segment may be called a Vector quantity. For this reason, velocity, acceleration, force, motion or displacement are called Vector quantities. A simple illustration - 50 feet due north. Fifty feet is the length or magnitude and due north is the direction.

2. Scalar. When units of measurements have been chosen, a scalar is represented by a real number and so is subject to all the laws of ordinary Algebra. Thus mass, time, density, coordinates, are scalars. The number 2 is a scalar.

A Vector, however, involving direction in addition to its numerical magnitude has an analysis peculiar to itself, the laws of which are to be derived.

3. Representation of a Vector. A Vector is represented graphically by an arrow of length equal to its magnitude, pointing in the assigned
direction from any point in space. An example: 3 miles due east.

\[ \overrightarrow{OA} \]

Figure 1.

The tail of the arrow, 0, is called the origin or the initial point; the head, A is called the end or the terminal point. In figures, we indicate the direction in which the Vector is drawn by an arrow.

We shall distinguish a Vector from a Scalar by placing a line or dash over the Vector symbol. So that if \( \overrightarrow{\text{a}} \) denotes a Vector, then a is its i-component.

Vectors are sometimes read with two letters. In Figure 1, the Vector \( \overrightarrow{OA} \) denotes the vector beginning at 0, ending at A, and pointing in the direction from 0 to A.

\[ \overrightarrow{OA} \]

Figure 2.

4. Equality of Vectors. Two vectors are equal if the line segments defining them are parallel or coincident, and their lengths and directions are the same. Thus, in Figure 2, \( \overrightarrow{OA} \) and \( \overrightarrow{OA'} \) are equal.

5. Negative Vector. The symbol \( -\overrightarrow{\text{a}} \), in Figure 2, is used to represent a vector having the length of \( \overrightarrow{OA} \) but the opposite direction. It is thus defined as the negative vector of \( \overrightarrow{\text{a}} \) and is written \( -\overrightarrow{\text{a}} \).
6. **The Unit Vector.** A unit vector will be denoted by adding the suffix 1 to the symbol representing the vector. This suffix tells us that its magnitude is unity.

The vector \( \bar{a} \) then may be considered as one \( a \) times as long as its unit vector \( \bar{a}_1 \), and hence, we may write
\[
\bar{a} = a \bar{a}_1 \quad \text{or} \quad \bar{a} = \bar{a}_0 \bar{a}_1
\]

7. **Magnitude of a Vector.** Associated with a vector \( \bar{a} \) is a positive scalar equal to its length, magnitude, or modulus. We shall represent this by the notation \( |\bar{a}| \). If then \( a \) is the length of the vector \( \bar{a} \),
\[
a = |\bar{a}| \quad \text{or} \quad |\bar{a}| = a
\]
The magnitude of a vector \( \bar{a} \) will be sometimes denoted by adding the subscript 0 to \( \bar{a} \) thus
\[
\bar{a}_0 = a
\]

8. **A Null Vector.** A null vector or a zero vector is one whose modulus is zero.

If \( \bar{a}_0 = 0 \),
the vector is a null vector and its direction is undefined.

9. **Collinear Vectors.** Parallel vectors regardless of their magnitude, size or tension are said to be collinear.

10. **Reciprocal Vectors.** The vector parallel to \( \bar{a} \), but whose length is the reciprocal of the length of \( \bar{a} \), is said to be the reciprocal of \( \bar{a} \). This is illustrated in Figure 3.

\[\begin{align*}
\bar{a} &= \delta \bar{a}_1 \\
\bar{a} &= \frac{1}{\delta} \bar{a}_1
\end{align*}\]

**Figure 3.**
So that if
\[ \bar{a} = a \bar{a}, \]
then
\[ \frac{1}{\bar{a}} = \frac{1}{a} = \frac{a_1}{a}. \]
CHAPTER III

COMPOSITION OF VECTORS

1. Addition of Vectors. Given any two vectors \( \vec{a} \) and \( \vec{b} \), we can construct a vector equal to the sum of vectors \( \vec{a} \) and \( \vec{b} \).

Given vectors

\[
\begin{align*}
\vec{a} & \quad \vec{b} \\
\end{align*}
\]

To construct a vector equal to their sum,

\[
\begin{align*}
\vec{a} + \vec{b} & = \vec{c} \\
\end{align*}
\]

Since vectors are free and are looked upon as operators which move from one place to another, we can slide them if they do not have the same origin, providing we do not alter their direction.

To obtain graphically the sum of the two vectors \( \vec{a} \) and \( \vec{b} \), draw a parallel to the given vector \( \vec{a} \); draw \( \vec{b} \) starting from the end of \( \vec{a} \) parallel to the given vector \( \vec{b} \); the vector joining the origin of \( \vec{a} \) with the end of \( \vec{b} \) is the sum indicated in Figure 4, by the notation \( \vec{a} + \vec{b} \). Now draw \( \vec{a} \) starting from the end of \( \vec{b} \); the vector joining the origin of \( \vec{b} \) to the end of \( \vec{a} \) is the sum in question \( (\vec{b} + \vec{a}) \). In other words, it is the diagonal of the parallelogram of which the two vectors \( \vec{a} \) and \( \vec{b} \) are the adjacent sides.
This is known as the parallelogram law of adding vectors.

\[ \vec{a} + \vec{b} = \vec{b} + \vec{a} \]  

(5)

This is called the commutative law of addition.

Similarly the sum of three or more vectors is obtained by constructing a polygon having those vectors as consecutive sides, and drawing a vector from the initial point of the first vector to the terminal point of the last.

Figure 5.

\[ \vec{a} + \vec{b} + \vec{c} = (\vec{a} + \vec{b}) + \vec{c} = (\vec{b} + \vec{c}) + \vec{a} = (\vec{c} + \vec{a}) + \vec{b} \]  

(6)

The sum is independent of the way its addends are associated into groups. This is called the associative law of addition.

2. Subtraction of Vectors. To subtract one vector from another, add to the first (\( \vec{a} \)) the second vector reversed (\(-\vec{b}\)).

Figure 6.

If two vectors \( \vec{a} \) and \( \vec{b} \) are drawn from the same origin (Figure 6) the difference \( \vec{a} - \vec{b} \) is defined as the vector extending from the end of \( \vec{b} \) to the end of \( \vec{a} \).
Figure 6 also shows that
\[ \overrightarrow{b} \neq (\overrightarrow{a} - \overrightarrow{b}) = \overrightarrow{a} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (7) \]

3. Components of a Vector. From addition and subtraction of vectors, it is clear that any vector \( \overrightarrow{q} \) may be considered as the sum of any number of component vectors, which when joined end to end, the first one begins at the origin of \( \overrightarrow{q} \) and the last one ends at the terminus of \( \overrightarrow{q} \). Thus
\[ \overrightarrow{q} = \overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} + \overrightarrow{d} + \overrightarrow{e} + \overrightarrow{f} + \overrightarrow{g}, \]
where \( \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{d}, \overrightarrow{e}, \overrightarrow{f}, \) and \( \overrightarrow{g} \) (Figure 7) are called component vectors of the \( \overrightarrow{q} \). These vectors need not lie in one plane. Vectors, all of which lie in or parallel to the same plane, are said to be coplanar.

![Figure 7.](image)

It is often convenient to decompose a vector into two or three components at right angles to each other; two in case all the vectors under consideration are coplanar; three when they are not coplanar.

4. The Unit Vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \). If vectors are coplanar, the unit vectors along the \( x- \), \( y- \) axes with origin at 0, are \( \mathbf{i} \) and \( \mathbf{j} \) respectively.

![Figure 8.](image)
In \( \vec{I} \) and \( \vec{J} \), \( x \) and \( y \) are the \( \vec{I} \) and \( \vec{J} \) components.

If vectors are non-coplanar, they can be referred to the Cartesian system of axes. The three unit vectors along the \( x-, y-, z- \) axes are called \( \vec{I}, \vec{J}, \vec{K} \) respectively.

Let \( \vec{F} \) be any vector equivalent to a vector \( \overrightarrow{OA} \) along \( \overrightarrow{OX} \), plus a vector \( \overrightarrow{AB} \) along \( \overrightarrow{OZ} \), plus a vector \( \overrightarrow{BC} \) along \( \overrightarrow{OY} \).

Figure 9.

In \( \vec{I}, \vec{J}, \vec{K} \), \( z \) is the magnitude and \( \vec{K} \) the direction; in \( \vec{J} \), \( y \) is the magnitude and \( \vec{J} \) the direction; in \( \vec{I} \), \( x \) is the magnitude and \( \vec{I} \) the direction.

If \( x, y, z \) denote the magnitude of the vectors respectively, we may write for any vector \( \vec{F} \) the magnitude of whose components are \( x, y, z \).

\[
\vec{F} = x \vec{I} + y \vec{J} + z \vec{K}.
\]

This vector \( \vec{F} \) is often used to represent the three projections of \( \vec{F} \) along the three axes respectively.

Let \( \alpha, \beta, \gamma \) be the direction angles of any vector parallel to \( \overrightarrow{OC} \).

Then

\[
\begin{align*}
x &= r \cos \alpha \ \\
y &= r \cos \beta \ \\
z &= r \cos \gamma
\end{align*}
\]
This decomposition of a vector into three rectangular components is the connecting link between two or three dimensional Cartesian and Vector Analysis respectively.

If two vectors are given

\[ \vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k} \]
and

\[ \vec{b} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k} \]

their sum is

\[ (\vec{a} + \vec{b}) = (a_x + b_x) \hat{i} + (a_y + b_y) \hat{j} + (a_z + b_z) \hat{k}. \]

This process of finding the sum may be extended to any number of vectors and shows that the components of the sum of n vectors are equal to the sums of the components of n vectors.

Symbolically, we may write

\[ \sum_{i=1}^{n} \vec{a}_i = \sum_{i=1}^{n} a_i \hat{i} \]
\[ \sum_{i=1}^{n} \vec{b}_i = \sum_{i=1}^{n} b_i \hat{j} \]
\[ \sum_{i=1}^{n} \vec{c}_i = \sum_{i=1}^{n} c_i \hat{k} \]

To obtain the rectangular components of a unit vector, equations (9) are used, so that

\[ \vec{r} = r (\hat{i} \cos \alpha \hat{i} + \hat{j} \cos \beta \hat{j} + \hat{k} \cos \gamma). \]

If we divide by r, we get

\[ \vec{F} = \vec{r} \]
\[ \hat{i} \cos \alpha \hat{i} + \hat{j} \cos \beta \hat{j} + \hat{k} \cos \gamma. \]

Therefore, the magnitude of the rectangular components of a unit vector are always its direction cosines.
CHAPTER IV

SCALAR AND VECTOR PRODUCTS OF TWO VECTORS

1. The Scalar or Dot Product. The scalar or dot product of two vectors \( \vec{a} \) and \( \vec{b} \), denoted by \( \vec{a} \cdot \vec{b} \), \( S \vec{a} \cdot \vec{b} \), \( \vec{a} \cdot \vec{b} \) or \( (\vec{a} \cdot \vec{b}) \) by various writers, is a scalar defined by the equation

\[
\vec{a} \cdot \vec{b} = ab \cos (\vec{a} \cdot \vec{b}) = b \cdot \vec{a}
\]

where \( (\vec{a} \cdot \vec{b}) \) is the angle between the two vectors.

The scalar or dot product is commutative.

\[
\vec{a} \cdot \vec{b} = ab \cos (\vec{a} \cdot \vec{b}) = \vec{b} \cdot \vec{a}
\]

Figure 10

\[
\vec{a} \cdot \vec{b} = ab \cos (\vec{a} \cdot \vec{b}) = \vec{b} \cdot \vec{a}
\]

shows that the scalar product may be looked upon as the product of the length of one of the two vectors multiplied by the projection of the other upon it,

or \( \vec{a} x \vec{b} = \vec{b} x \vec{a} \).

Evidently, if the two vectors \( \vec{a} \) and \( \vec{b} \) are perpendicular to each other \( \cos (\vec{a} \cdot \vec{b}) = 0 \) and their scalar product is zero.

Or, if

\[
\vec{a} \cdot \vec{b} = 0, \text{ then } \vec{a} \perp \vec{b} \quad \quad \quad \quad \quad \quad \quad \quad (15)
\]

If \( \vec{a} \) and \( \vec{b} \) are parallel vectors, \( \cos (\vec{a} \cdot \vec{b}) = 1 \) and

\[
\vec{a} \cdot \vec{b} = ab
\]
and in particular if \( \overrightarrow{b} = \overrightarrow{a} \),
\[
\overrightarrow{a} \cdot \overrightarrow{a} = a^2
\]
The scalar product of a vector into itself is often written as the square of the vector, thus,
\[
\overrightarrow{a} \cdot \overrightarrow{a} = a^2
\]
In general, to obtain the magnitude of a vectorial expression, it is only necessary to square it, and the result is the square of its absolute value, magnitude or size.

2. Laws of the Scalar Product. The scalar product obeys the ordinary laws of multiplication. Consider the two vectors \( \overrightarrow{c} \) and \( \overrightarrow{d} \) as well as their sum \( (\overrightarrow{c} + \overrightarrow{d}) \). Consider also their projections upon any other vector \( \overrightarrow{b} \).

![Diagram](image)

The projection of \( \overrightarrow{c} \) on \( \overrightarrow{b} \) is \( CE \); the projection of \( \overrightarrow{d} \) on \( \overrightarrow{b} \) is \( EF \); the projection of \( (\overrightarrow{c} + \overrightarrow{d}) \) on \( \overrightarrow{b} \) is \( CF \); hence
\[
\overrightarrow{c} \cdot \overrightarrow{b} + \overrightarrow{d} \cdot \overrightarrow{b} = (\overrightarrow{c} + \overrightarrow{d}) \cdot \overrightarrow{b} = \overrightarrow{b} \cdot (\overrightarrow{c} + \overrightarrow{d}) \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (16)
\]
This result is easily extended to the scalar product of the sum of any number of vectors.

Observe from the above equation, that scalar multiplication or the dot product is also distributive; that is, it obeys the distributive law of multiplication with respect to addition.
3. Application of the Scalar Product to Unit Vectors \( \hat{I}, \hat{J}, \hat{K} \). The application of the scalar product of two vectors to the unit vectors \( \hat{I}, \hat{J}, \) and \( \hat{K} \) is of great importance, and gives

\[
\begin{align*}
\hat{I} \cdot \hat{I} &= \hat{J} \cdot \hat{J} = \hat{K} \cdot \hat{K} = 1 \\
\hat{I} \cdot \hat{J} &= \hat{J} \cdot \hat{K} = \hat{K} \cdot \hat{I} = 0
\end{align*}
\]

Let us consider two vectors \( \vec{a} \) and \( \vec{b} \) given in terms of their components along the \( x-, y-, z- \) axes:

\[
\vec{a} = a_x \hat{I} + a_y \hat{J} + a_z \hat{K},
\]

and

\[
\vec{b} = b_x \hat{I} + b_y \hat{J} + b_z \hat{K},
\]

then by (16) and (17)

\[
\begin{align*}
\vec{a} \cdot \vec{b} &= (a_x \hat{I} + a_y \hat{J} + a_z \hat{K}) \cdot (b_x \hat{I} + b_y \hat{J} + b_z \hat{K}) \\
\vec{a} \cdot \vec{b} &= a_x b_x + a_y b_y + a_z b_z
\end{align*}
\]

Consider any two vectors \( \vec{a} \) and \( \vec{b} \). We wish to find the magnitude of the sum of the two vectors and the angle between them. The square of the magnitude of the sum is found by dotting the sum of the two vectors into itself.

In triangle ABC

\[
\vec{c} = \vec{a} + \vec{b}
\]

Figure 12.
Squaring to find its magnitude, we obtain
\[ \overrightarrow{c} \cdot \overrightarrow{c} = c^2 = (\overrightarrow{a} \cdot \overrightarrow{b}) \cdot (\overrightarrow{a} \cdot \overrightarrow{b}) = \overrightarrow{a} \cdot \overrightarrow{b} \cdot 2\overrightarrow{a} \cdot \overrightarrow{b} \cdot \overrightarrow{b} \cdot \overrightarrow{b} \]

Or
\[ c^2 = a^2 + 2ab \cos (\theta) \cdot b^2 \]
\[ \therefore c^2 = \text{the square of the magnitude of the sum, } \overrightarrow{a} \pm \overrightarrow{b}. \]

Or
\[ c^2 = a^2 - 2ab \cos (\theta) \cdot b^2 \]

Where \( \theta \) is the supplement to the angle between \( \overrightarrow{a} \) and \( \overrightarrow{b} \). This is the Law of Cosines.

**4. The Vector or Cross Product.** The vector product of two vectors \( \overrightarrow{a} \) and \( \overrightarrow{b} \) is a vector, written \( \overrightarrow{a} \times \overrightarrow{b} \) (in distinction from \( \overrightarrow{a} \cdot \overrightarrow{b} \), the dot product), also \( \forall \overrightarrow{a} \times \overrightarrow{b} \) or \( [\overrightarrow{a} \overrightarrow{b}] \) by different authors, and is defined by the equation.
\[ \overrightarrow{a} \times \overrightarrow{b} = E \, ab \, \sin (\overrightarrow{a} \overrightarrow{b}) = -\overrightarrow{b} \times \overrightarrow{a} \]

where \( E \) is a unit vector, normal to the plane of \( \overrightarrow{a} \) and \( \overrightarrow{b} \) and so directed that as you turn the first named vector \( \overrightarrow{a} \) into the second one \( \overrightarrow{b} \), \( E \) points in the direction that a right-handed screw (cork-screw) would progress if turned in this same manner.

\[ E \, ab \, \sin (\overrightarrow{a} \overrightarrow{b}) = \overrightarrow{a} \times \overrightarrow{b} \]

**Figure 13.**
In other words, \( \mathbf{a} \times \mathbf{b} \) is a vector perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \) and whose magnitude may be represented by the area of a parallelogram of which \( \mathbf{a} \) and \( \mathbf{b} \) are the adjacent sides.

If the vector \( \mathbf{b} \) came first instead of the vector \( \mathbf{a} \), in the product, the only difference would be in the reversal of the direction of \( \mathbf{E} \), so that

\[
\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}
\]

If \( \mathbf{a} \) and \( \mathbf{b} \) be finite vectors and

\[
\mathbf{a} \times \mathbf{b} = 0,
\]

then \( \mathbf{a} \parallel \mathbf{b} \)

The sine of their included angle must be zero. This then is the condition for parallelism of the two vectors \( \mathbf{a} \) and \( \mathbf{b} \).

Since any vector is parallel to itself,

\[
\mathbf{a} \times \mathbf{a} = 0.
\]

5. Application of Cross Product to Unit Vectors. Remembering that the unit vectors \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \) are mutually perpendicular, it follows from the definition \( \mathbf{a} \times \mathbf{b} \) that (Figure 11.)

\[
\begin{align*}
\mathbf{j} \times \mathbf{k} &= \mathbf{i} = -\mathbf{k} \times \mathbf{j} \\
\mathbf{k} \times \mathbf{i} &= \mathbf{j} = -\mathbf{i} \times \mathbf{k} \\
\mathbf{i} \times \mathbf{j} &= \mathbf{k} = -\mathbf{j} \times \mathbf{i}
\end{align*}
\]
We also have, by (20)
\[ \bar{1} \times \bar{1} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = 0. \]

The cross product of two vectors can be expressed in terms of their components. Thus if
\[ \bar{a} = a_1 \bar{i} + a_2 \bar{j} + a_3 \bar{k}, \]
and
\[ \bar{b} = b_1 \bar{i} + b_2 \bar{j} + b_3 \bar{k}, \]
by expanding and using (21), we get
\[ \bar{a} \times \bar{b} = (a_1 \bar{i} + a_2 \bar{j} + a_3 \bar{k}) \times (b_1 \bar{i} + b_2 \bar{j} + b_3 \bar{k}). \]

Grouping terms with like unit vectors, we obtain
\[ \bar{a} \times \bar{b} = (a_1 b_3 - a_3 b_1) \bar{i} + (a_2 b_1 - a_1 b_2) \bar{j} + (a_3 b_2 - a_2 b_3) \bar{k}. \]

6. Cartesian Expansion of the Vector Product. As a determinant, this can be written
\[ \bar{a} \times \bar{b} = \begin{vmatrix} 1 & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \]

The cross product is not commutative.
CHAPTER V
VECTOR AND SCALAR PRODUCTS INVOLVING THREE VECTORS

1. Possible Combinations of Three Vectors. From the three vectors \( \vec{a} \), \( \vec{b} \), and \( \vec{c} \), the following combinations may be derived:

   1. \( \vec{a} \cdot (\vec{b} \times \vec{c}) \) (a vector)
   2. \( \vec{a} \cdot (\vec{b} \times \vec{c}) \) (a scalar)
   3. \( \vec{a} \times (\vec{b} \times \vec{c}) \) (a vector)
   4. \( \vec{a} \cdot (\vec{b} \times \vec{c}) \) (not defined)
   5. \( \vec{a} \cdot (\vec{b} \times \vec{c}) \) (absurd)
   6. \( \vec{a} \times (\vec{b} \times \vec{c}) \) (absurd)

Of these six expressions, 5 and 6 are meaningless and absurd, because they are the scalar product and vector product, respectively, of a vector \( \vec{a} \) and a scalar \( (\vec{b} \cdot \vec{c}) \) and such products require a vector on each side of the dot or cross. As to 4, since no definition of the product of two vectors without a dot or a cross has been made, it is yet meaningless. We shall consider in detail the three remaining triple products. The first one of these, \( \vec{a} \cdot (\vec{b} \times \vec{c}) \) is simply the vector \( \vec{a} \) multiplied by the scalar quantity \( (\vec{b} \cdot \vec{c}) \) and is a vector in the same direction as \( \vec{a} \), but \( \vec{b} \cdot \vec{c} \) times longer. This triple product, then, offers no new difficulties, and means

\[
\vec{a} \cdot (\vec{b} \times \vec{c}) = |\vec{a}| \cdot |\vec{b} \times \vec{c}| = |\vec{a}| \cdot |\vec{b}| \cdot |\vec{c}| \sin \theta \\
= |\vec{a}| \cdot |\vec{b}| \cdot |\vec{c}| \cos \theta \\
= \vec{a} \times \vec{b} \cdot \vec{c}.
\]

2. The Triple Scalar Product \( V = \vec{a} \cdot (\vec{b} \times \vec{c}) \). This product is a scalar and represents the volume of a parallelepiped of which the three contemnuous edges are \( \vec{a} \), \( \vec{b} \), and \( \vec{c} \). This is seen to be the case as \( \vec{b} \times \vec{c} \) is the area of the base represented by a vector \( \vec{b} \times \vec{c} \) to this base; the scalar product of \( \vec{a} \) and the vector \( \vec{b} \times \vec{c} \) will be this area multiplied by the projection of the slant height \( \vec{a} \) along it, or in other words, the volume. This volume, \( V \), may be obtained by forming the vector products of any two or three vectors \( \vec{a} \), \( \vec{b} \) and \( \vec{c} \) (thus giving the area of one of the faces) and forming the scalar product of the vector-area with the remaining third
vector, it follows that
\[ V = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}). \]

If the vectors \((\mathbf{c} \times \mathbf{a}), (\mathbf{a} \times \mathbf{b}),\) and \((\mathbf{b} \times \mathbf{c})\) are taken so that they form an acute angle with \(\mathbf{b}, \mathbf{c},\) and \(\mathbf{a},\) respectively; then the volume is to be considered positive, the cosine term in the scalar product being positive. (Figure 15). Otherwise, the volume is to be considered negative. The inversion of the factors in the vector products should change the sign by (19), so that we have

\[ (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{a}) = - \mathbf{E} \cdot (\mathbf{a} \times \mathbf{c}) = - (\mathbf{a} \times \mathbf{c}) \cdot \mathbf{E} \]

By a consideration of these equalities the following laws may be seen to hold:

1. The sign of the scalar triple product is unchanged as long as the cycli-
cal order of the factors is unchanged.

2. For every change of cyclical order a minus sign is introduced.

3. The dot and cross may be interchanged ad libitum.

The equalities (21) are called by Heaviside the Parallelopiped Law.
3. Cartesian Expansion of Triple Scalar Product $V = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

The product may be written in terms of the components of its vectors along the $X-$, $Y-$, $Z-$ axes in the form of a determinant.

If

$$\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k},$$

and

$$\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k},$$

and

$$\mathbf{c} = c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k},$$

the triple scalar product is equal to the determinant,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

For

$$\begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3 \\
\end{vmatrix}$$

The triple scalar product can be written as the minors of $a_1$, $a_2$, $a_3$, respectively.

4. The Triple Vector Product $\mathbf{q} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. The triple product

$$\mathbf{q} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

is a vector. In this expression, the parenthesis, or some separating symbol, is necessary, as $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. The sign of this product changes every time the order of the factors $\mathbf{a}$ and $(\mathbf{b} \times \mathbf{c})$ is changed in $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, or whenever the order of the factors $\mathbf{b}$ and $\mathbf{c}$ is changed in $(\mathbf{b} \times \mathbf{c})$. The vector product being always 1 to both of its components, $\mathbf{q}$ is 1 to $\mathbf{a}$ as well as to $\mathbf{b} \times \mathbf{c}$, hence

$$\mathbf{q} \cdot \mathbf{a} = 0$$

and $\mathbf{q} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.

Equation (26) shows that $\mathbf{q}$ lies in the same plane as $\mathbf{b}$ and $\mathbf{c}$, either by the condition that the three non-parallel vectors should lie in a plane is that

$$[\mathbf{a} \, \mathbf{b} \, \mathbf{c}] = 0$$

or by seeing that it is 1 to a line which is itself 1 to
5. Expansion and Proof of Triple Vector Product. The triple vector product \( \mathbf{q} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) can be reduced to simpler forms by means of the equation,

\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} (\mathbf{a} \cdot \mathbf{c}) - \mathbf{c} (\mathbf{a} \cdot \mathbf{b}) \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (27)
\]

Proof:

Express vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) in terms of the unit vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \); expand both sides and show that the two sides are equal.

Given

\[
\begin{align*}
\mathbf{a} &= a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \\
\mathbf{b} &= b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} \\
\mathbf{c} &= c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k}
\end{align*}
\]

\[
\begin{align*}
\mathbf{b} (\mathbf{a} \cdot \mathbf{c}) &= (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) (c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k}) \\
&= a_1 c_1 \mathbf{i} \mathbf{i} + a_1 c_2 \mathbf{i} \mathbf{j} + a_1 c_3 \mathbf{i} \mathbf{k} \\
&\quad + a_2 c_1 \mathbf{j} \mathbf{i} + a_2 c_2 \mathbf{j} \mathbf{j} + a_2 c_3 \mathbf{j} \mathbf{k} \\
&\quad + a_3 c_1 \mathbf{k} \mathbf{i} + a_3 c_2 \mathbf{k} \mathbf{j} + a_3 c_3 \mathbf{k} \mathbf{k}
\end{align*}
\]

\[
\begin{align*}
\mathbf{c} (\mathbf{a} \cdot \mathbf{b}) &= -a_1 (a_1 b_1 \mathbf{i} + a_2 b_2 \mathbf{j} + a_3 b_3 \mathbf{k}) \\
&\quad -a_2 (a_1 b_2 \mathbf{i} + a_2 b_2 \mathbf{j} + a_3 b_3 \mathbf{k}) \\
&\quad -a_3 (a_1 b_3 \mathbf{i} + a_2 b_2 \mathbf{j} + a_3 b_3 \mathbf{k})
\end{align*}
\]

Therefore

\[
\begin{align*}
\mathbf{b} (\mathbf{a} \cdot \mathbf{c}) &= a_1 c_1 \mathbf{i} \mathbf{i} + a_1 c_2 \mathbf{i} \mathbf{j} + a_1 c_3 \mathbf{i} \mathbf{k} \\
&\quad + a_2 c_1 \mathbf{j} \mathbf{i} + a_2 c_2 \mathbf{j} \mathbf{j} + a_2 c_3 \mathbf{j} \mathbf{k} \\
&\quad + a_3 c_1 \mathbf{k} \mathbf{i} + a_3 c_2 \mathbf{k} \mathbf{j} + a_3 c_3 \mathbf{k} \mathbf{k}
\end{align*}
\]

\[
\begin{align*}
\mathbf{c} (\mathbf{a} \cdot \mathbf{b}) &= -a_1 (a_1 b_1 \mathbf{i} + a_2 b_2 \mathbf{j} + a_3 b_3 \mathbf{k}) \\
&\quad -a_2 (a_1 b_2 \mathbf{i} + a_2 b_2 \mathbf{j} + a_3 b_3 \mathbf{k}) \\
&\quad -a_3 (a_1 b_3 \mathbf{i} + a_2 b_2 \mathbf{j} + a_3 b_3 \mathbf{k})
\end{align*}
\]

Grouping coefficients of like unit vectors, we obtain

\[
\begin{align*}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{b} (\mathbf{a} \cdot \mathbf{c}) - \mathbf{c} (\mathbf{a} \cdot \mathbf{b}) \\
&= \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
(a_2 b_1 - a_1 b_2) & a_3 (b_3 c_1 - b_1 c_3) & a_1 (b_1 c_2 - b_2 c_1) \\
(b_2 c_1 - b_1 c_2) & a_3 (b_3 c_1 - b_1 c_3) & a_1 (b_1 c_2 - b_2 c_1) \\
(c_2 b_1 - b_1 c_2) & a_3 (b_3 c_1 - b_1 c_3) & a_1 (b_1 c_2 - b_2 c_1)
\end{vmatrix}
\end{align*}
\]

As a determinant, we may write

\[
\begin{vmatrix}
\mathbf{a} \\
\mathbf{b} \\
\mathbf{c}
\end{vmatrix} = \begin{vmatrix}
a_1 & a_2 & a_3 \\
b_2 & b_3 & b_1 \\
c_2 & c_3 & c_1
\end{vmatrix} \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k}
\end{vmatrix}
\]
Hence,

\[ \overline{a} \times (\overline{b} \times \overline{c}) = \overline{b} \ (\overline{a} \cdot \overline{c}) - \overline{c} \ (\overline{a} \cdot \overline{b}) \]

Q.E.D.
1. The equation
\[(\vec{r} - \vec{a})_0 = (\vec{r} - \vec{b})_0\]
represents the plane bisecting at right angles the line A B.

Figure 16.

Given the plane MN bisecting the line AB at point R making right angles.

To prove: \[(\vec{r} - \vec{a})_0 = (\vec{r} - \vec{b})_0\].

Proof:

Let 0 be any point in space and P any point on plane MN.

Draw \(\overrightarrow{CA}\), \(\overrightarrow{CP}\), and \(\overrightarrow{CB}\).

Let \(\overrightarrow{CA} = \overrightarrow{a}\).
\(\overrightarrow{CB} = \overrightarrow{b}\).
\(\overrightarrow{CP} = \vec{r}\), the variable vector.
Then,
$$\overrightarrow{r} - \overrightarrow{a}$$ is the vector constructed from the end of \(\overrightarrow{a}\) to the end of \(\overrightarrow{r}\). The origin of \(\overrightarrow{a}\) and \(\overrightarrow{r}\) is at point 0.

By similar construction on vectors \(\overrightarrow{b}\) and \(\overrightarrow{r}\), we obtain \(\overrightarrow{r} - \overrightarrow{b}\).

Plane \(MN\) is the locus of all points equidistant from \(A\) and \(B\). \(P\) is a point on plane \(MN\).

Hence,
$$\left| \overrightarrow{r} - \overrightarrow{a} \right| = \left| \overrightarrow{r} - \overrightarrow{b} \right|.$$

The magnitude or length of these two vectors \((\overrightarrow{r} - \overrightarrow{a})\) and \((\overrightarrow{r} - \overrightarrow{b})\) may be represented by:
$$(\overrightarrow{r} - \overrightarrow{a})_0 = \left| \overrightarrow{r} - \overrightarrow{a} \right|,$$
and
$$(\overrightarrow{r} - \overrightarrow{b})_0 = \left| \overrightarrow{r} - \overrightarrow{b} \right|.$$

We refer to Chapter II, Equation (1). Having proved the vectors equal in magnitude, I conclude that the length, magnitude or size of the vectors are equal.

That is
$$(\overrightarrow{r} - \overrightarrow{a})_0 = (\overrightarrow{r} - \overrightarrow{b})_0.$$

Q.E.D.

2. The lines which join one vertex of a parallelogram to the middle points of the opposite sides locate the trisection points on the diagonal which does not go through the vertex.

![Diagram](image-url)

Figure 17.
Given the parallelogram $OBCD$, $E$ the midpoint of $CD$, $F$ the midpoint of $DC$, $CC$ the diagonal; $BE$ meeting $OC$ at $R$, and $BF$ meeting $OC$ at $S$.

To prove $BE$ trisects $OC$, and $BF$ trisects $OC$, i.e., $R$ is a trisection point of $OC$ and $S$ is a trisection point of $OC$. This amounts to proving

$OR = \frac{1}{3} OC$, and $CS = \frac{1}{3} OC$.

Proof:

Let $\overrightarrow{CD} = a$, and $\overrightarrow{OB} = b$.

Then

$\overrightarrow{OE} = \frac{1}{2} a$.

Also

$\overrightarrow{EB} = b - \frac{1}{2} a$.

and

$\overrightarrow{EB} = S (b - \frac{1}{2} a)$.

$\overrightarrow{OC} = (a + b)$.

See composition of vectors - Chapter III.

Now

$\overrightarrow{OR} = \overrightarrow{OE} + \overrightarrow{ER} = \frac{1}{2} a + S (b - \frac{1}{2} a)$,

where $S$ is an unknown scalar representing some part or portion of $EB$.

Also

$\overrightarrow{OR} = t (a + b)$,

where $t$ is an unknown scalar representing some part or portion of $OC$.

Quantities which are equal to the same quantity or to equal quantities are equal to one another.

Hence,

$\frac{1}{2} a + S (b - \frac{1}{2} a) = t (a + b)$.

Multiplying, we obtain

$1/2 a + S b - 1/2 S a = t a + t b$.

By collecting terms, it follows that

$S b - 1/2 (1 - S) a = t a + t b$. 
The scalar coefficients of \( \bar{a} \) on both sides of the equation are equal.

The scalar coefficient of \( a \) on the left-hand side of the equation is \( \frac{1}{2} (1-S) \). The scalar coefficient of \( \bar{a} \) on the right-hand side of the equation is \( t \).

\[ \frac{1}{2} (1-S) = t. \]

The scalar coefficients of \( \bar{b} \) on both sides of the equation are equal.

The scalar coefficient of \( \bar{b} \) on the left-hand side of the equation is \( S \). The scalar coefficient of \( \bar{b} \) on the right-hand side of the equation is \( t \).

\[ S = t. \]

We now have two linear non-homogeneous equations in two unknowns (\( S \) and \( t \)).

Solving the equations simultaneously, we get

\[ S = t; \quad \frac{1}{2} (1-S) = t. \]

Substituting the value of \( S \) from equation (1) in equation (2), we get

\[ \frac{1}{2} (1-t) = t. \]

Multiplying by 2, we obtain

\[ 1 - t = 2t. \]

By adding \( t \) to both sides of the equation, we get

\[ 1 = 3t. \]

Dividing both sides of the equation by 3, we obtain

\[ \frac{1}{3} = t. \]

Since

\[ S = t, \]

and

\[ t = \frac{1}{3} \]

then
S = 1/3.

Substituting the value of t in the equation
\[ \overrightarrow{OR} = t(\overrightarrow{a} + \overrightarrow{b}) \]
we get \[ \overrightarrow{OR} = \frac{1}{3} (\overrightarrow{a} \cdot \overrightarrow{b}). \]

But

\( (\overrightarrow{a} \cdot \overrightarrow{b}) \) is the diagonal OC,

\[ \therefore \text{I conclude that} \]

OR = \( \frac{1}{3} \) OC.

Similarly, it can be shown that

CS = \( \frac{1}{3} \) OC

Q.E.D

3. The bisectors of the angles of any triangle meet in a point. Before I can prove this I must first prove the Lemma,

The equation of the bisector of the angle formed by two vectors \( \overrightarrow{a} \) and \( \overrightarrow{b} \) from the same origin o is \( \overrightarrow{r} = y (\overrightarrow{a} \cdot \overrightarrow{b}) \).

Proof of Lemma:

Employ unit vectors along two of the sides as independent vectors. The bisector is then \( \overrightarrow{a} \cdot \overrightarrow{b} \).

Let OP be the bisector of the angle formed by \( \overrightarrow{a} \) and \( \overrightarrow{b} \) with P any point on it. Through P draw PQ \( \parallel \) OB meeting OA or OA produced at Q.
Through A draw a line \( \parallel \overline{CD} \) meeting \( \overline{OP} \) at \( C \).

\[
\frac{\overline{AC}}{\overline{CB}} = k
\]

Multiplying by \( |\overline{OB}| \), we get

\[
|\overline{AC}| = k |\overline{OB}|
\]

\[
\overline{AC} = k \overline{OB} = k \overline{B}.
\]

\[
\angle BOC = \angle COA.
\]

for \( \angle BOC \) was bisected

\[
\angle BOC = \angle COA.
\]

If two parallels are cut by a transversal, the alt. int. \( \text{'s} \) are \( \text{=} \).

\[
\angle COA = \angle ACO.
\]

Quantities which are equal to the same quantity or to equal quantities are equal

\[
|\overline{AC}| = |\overline{OA}|.
\]

If the base \( l^2 \) of a triangle are \( = \) the sides opposite them are equal.

\[\triangle OAC \] is isosceles.

An isosceles triangle has two equal sides.

Since \( \overline{AC} = k \overline{B} \),

and

\[
\overline{AC} = \overline{a}.
\]

Then

\[
k \overline{B} = \overline{a}.
\]

Quantities which are equal to the same quantity or to equal quantities are equal.

The equation of the bisector \( \overline{OF} \) is

\[
\overline{r} = x (\overline{OC}),
\]

where \( x \) represents a scalar portion or part of \( \overline{OF} \).

\[
\overline{r} = x (\overline{OA} \neq \overline{AC}).
\]

See composition of vectors - Chapter III.
Refer to Chapter II - Topic 6, Page 5.

Factoring, we obtain

$$\vec{r} = x \left( a \vec{a} \neq \vec{b} \right) = x \left( a \vec{a}_1 \neq \vec{b}_1 \right).$$

Hence

$$r = y \left( a \vec{a}_1 \neq \vec{b}_1 \right),$$

where $y$ represents the product of $x \cdot a$.

Hence the Lemma is proved.

Now let us prove the theorem.

Given $BQ$, $AP$, and $CR$, bisectors of the $\angle B$, $\angle A$, and $\angle C$ of triangle $BAC$ respectively. To prove $BQ$, $AP$ and $CR$ meet at point $I$.

By the Lemma:

With reference to vector origin $A$, equation of $AP$ is $\vec{r} = x \left( \vec{a}_1 - \vec{b}_1 \right)$; with reference to vector origin at $O$, equation of $AP$ is

$$\vec{r} = \vec{s} \neq x \left( \vec{a}_1 - \vec{b}_1 \right).$$

Figure 19.
With reference to vector origin $B$, equation of $\overrightarrow{BQ}$ is $\overrightarrow{r} = y (\overrightarrow{a_1} - \overrightarrow{c_1})$; 
with reference to vector origin at $O$, equation of $\overrightarrow{BQ}$ is 
$$\overrightarrow{r} = t \neq y (\overrightarrow{a_1} - \overrightarrow{c_1}) \quad (2)$$

With reference to vector origin $C$, equation of $\overrightarrow{GR}$ is $\overrightarrow{r} = z (\overrightarrow{b_1} - \overrightarrow{a_1})$; 
with reference to vector origin at $O$, equation of $\overrightarrow{GR}$ is 
$$\overrightarrow{r} = \overrightarrow{u} \neq z (\overrightarrow{b_1} - \overrightarrow{a_1}) \quad (3)$$

Let $I$ be the intersection of $AP$ and $BQ$. Equating the right members of 
equations (1) and (2) we get 
$$\overrightarrow{s} \neq x (\overrightarrow{c_1} - \overrightarrow{b_1}) \neq t \neq y (\overrightarrow{a_1} - \overrightarrow{c_1})$$

Or, 
$$\overrightarrow{s} - \overrightarrow{t} \neq x (\overrightarrow{c_1} - \overrightarrow{b_1}) - y (\overrightarrow{a_1} - \overrightarrow{c_1}) = 0 \quad (4)$$

But 
$$\overrightarrow{s} \neq \overrightarrow{c} = \overrightarrow{t}$$

Transposing, we obtain 
$$\overrightarrow{s} - \overrightarrow{t} = -\overrightarrow{c} = c \overrightarrow{c_1} \quad (5)$$

Substituting (5) in equation (4): 
$$-c \overrightarrow{c_1} \neq x (\overrightarrow{c_1} - \overrightarrow{b_1}) - y (\overrightarrow{a_1} - \overrightarrow{c_1}) = 0 \quad (6)$$

But 
$$\overrightarrow{a} \neq \overrightarrow{b} \neq \overrightarrow{c} = 0,$$

or 
$$a \overrightarrow{a_1} \neq b \overrightarrow{b_1} \neq c \overrightarrow{c_1} = 0.$$ 

Transposing and dividing by $b$, we obtain 
$$\overrightarrow{b_1} = \frac{-a \overrightarrow{a_1} - c \overrightarrow{c_1}}{b} \quad (7)$$

By (6) and (7), it follows that 
$$-c \overrightarrow{c_1} \neq x (\overrightarrow{c_1} \neq a \overrightarrow{a_1} + c \overrightarrow{c_1} - y (\overrightarrow{a_1} - \overrightarrow{c_1}) = 0.$$ 

Multiplying by $b$, we get
By grouping, we obtain
\[-bc \neq x b \neq x a \neq x c \neq y b \neq y b \neq c = 0\]

By setting coefficients of $c_i$ and $a^1$ equal to zero, we obtain two linear equations
\[
\begin{cases}
a x - b y = 0 \\
-b c \neq b x \neq c x \neq y b = 0
\end{cases}
\]

We shall take the first equation of (8) and solve for $ax$.

\[a x - b y = 0.
\]

Transposing, we get
\[a x = b y.
\]

By substituting (9) in the second equation of (8) we obtain
\[-bc \neq b x \neq c x \neq a x = 0.
\]

Simplifying, it follows that
\[-bc \neq x (b \neq c \neq a) = 0.
\]

Or, by transposing, we obtain
\[x (a \neq b \neq c) = b c.
\]

Or,
\[x = \frac{bc}{a \neq b \neq c}.
\]

In equation (9),
\[a x = b y
\]
\[y = \frac{ax}{b}.
\]

Substituting the value of $x$ from equation (10), we obtain
\[y = \frac{abc}{b (a \neq b \neq c)}.
\]

Or,
\[y = \frac{ac}{(a \neq b \neq c)}.
\]
At the point of intersection of $\overline{EQ}$ and $\overline{OR}$, $r$ from (2) = $r$ from (3), i.e.,

$\overline{t} \neq y (\overline{a_1} - \overline{c_1}) = \overline{u} \neq z (\overline{b_1} - \overline{a_1})$,

Or

$\overline{t} - \overline{u} \neq y (\overline{a_1} - \overline{c_1}) - z (\overline{b_1} - \overline{a_1}) = 0$. (11)

But

$\overline{t} \neq \overline{a} = \overline{u}$.

Or

$\overline{t} - \overline{u} = - \overline{a} = - a \overline{a_1}$. (12)

Substituting (12) in equation (11):

$- a \overline{a_1} \neq y (\overline{a_1} - \overline{c_1}) - z (\overline{b_1} - \overline{a_1}) = 0$. (13)

Substituting the value of equation (7) in equation (13), we get

$- a \overline{a_1} \neq y (\overline{a_1} - \overline{c_1}) - z \left( - a \overline{a_1} - c \overline{c_1} - \overline{a_1} \right) = 0$

Multiplying by $b$, we obtain

$- a \overline{a_1} \neq b y \overline{a_1} - b y \overline{c_1} \neq a + \overline{a_1} - c + \overline{c_1} \neq b + \overline{a_1} = 0$

Grouping like terms, we get

$(- a b \neq b y \neq a s \neq b s) \overline{a_1} \neq (- b y \neq c s) \overline{c_1} = 0$

By setting coefficients of $\overline{a_1}$ and $\overline{c_1}$ equal to zero, we obtain two linear equations,

$(- a b \neq b y \neq c s = 0)$, $(- a b \neq b y \neq a s \neq b s = 0)$. (14)

From the first equation in (14)

$b y = c s$. (15)

Substituting (15) in the second equation in (14), we get

$- a b \neq c s \neq a s \neq b s = 0$

Factoring the last three terms, we obtain.
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\[-ab \neq a (a \neq b \neq c) = 0.\]

Solving for \(z\), we get

\[z \ (a \neq b \neq c) = ab\]

\[z = \frac{ab}{a \neq b \neq c} \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (16)\]

From (15)

\[b \ y = c \ z\]

\[y = \frac{c \ z}{b}\]

Substituting the value of (16) in the above equation, we get

\[y = \frac{abc}{b (a \neq b \neq c)}\]

Or

\[y = \frac{ac}{a \neq b \neq c}\]

Hence since the value of \(y\) for the point of intersection of \(BQ\) and \(CR\) is the same as the value of \(y\) for the point of intersection of \(AP\) and \(I\)

\(I\) is the intersection of \(AP\), \(BQ\), and \(CR\).

**Summary:**

\[AI = \frac{b \ c}{a \neq b \neq c}\]

\[BI = \frac{a \ c}{a \neq b \neq c}\]

\[CI = \frac{a \ b}{a \neq b \neq c}\]

Q.E.D.

4. The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides.

![Figure 20](image-url)
Given parallelogram $ABCD$ with diagonals $AC$ and $BD$.

To Prove: $(AC)^2 \neq (BD)^2 = (AB)^2 \neq (BC)^2 \neq (CD)^2 \neq (DA)^2$.

Proof:

Let $\overrightarrow{AB} = \overrightarrow{a}$, and 
$\overrightarrow{AD} = \overrightarrow{b}$.

But $\overrightarrow{BC} = \overrightarrow{a}$.

Also $\overrightarrow{DC} = \overrightarrow{b}$.

Then $\overrightarrow{AC} = \overrightarrow{a} \neq \overrightarrow{b}$.

And $\overrightarrow{BD} = \overrightarrow{a} - \overrightarrow{b}$.

Hence, we have 
$(AC)^2 \neq (BD)^2 = (\overrightarrow{a} \neq \overrightarrow{b})^2 \neq (\overrightarrow{a} - \overrightarrow{b})^2$.

Dotting each vector into itself, we obtain 
$$(\overrightarrow{a} \neq \overrightarrow{b}) \cdot (\overrightarrow{a} \neq \overrightarrow{b}) \neq (\overrightarrow{a} - \overrightarrow{b}) \cdot (\overrightarrow{a} - \overrightarrow{b})$$

$= (\overrightarrow{a} \cdot \overrightarrow{a} \neq \overrightarrow{a} \cdot \overrightarrow{b} \neq \overrightarrow{b} \cdot \overrightarrow{b}) \neq (\overrightarrow{a} \cdot \overrightarrow{a} - \overrightarrow{a} \cdot \overrightarrow{b} - \overrightarrow{b} \cdot \overrightarrow{b})$

$= (\overrightarrow{a}^2 \neq 2 \overrightarrow{a} \cdot \overrightarrow{b} \neq \overrightarrow{b}^2) \neq (\overrightarrow{a}^2 - 2 \overrightarrow{a} \cdot \overrightarrow{b} - \overrightarrow{b}^2) = 2\overrightarrow{a}^2 \neq 2\overrightarrow{b}^2$.

Hence, we conclude that 
$(AC)^2 \neq (BD)^2 = (AB)^2 \neq (BC)^2 \neq (CD)^2 \neq (DA)^2$.

Q.E.D.

5. Take equation of circle 
$$\overrightarrow{F}^2 - 2 \overrightarrow{a} \cdot \overrightarrow{F} = 0,$$

factor with $\overrightarrow{F}$ and interpret.
\[ \vec{r}^2 - 2 \vec{a} \cdot \vec{r} = 0. \]
\[ \vec{r} \cdot (\vec{r} - 2 \vec{a}) = 0, \]
represents the dot product of two vectors, \( \vec{F} \) and \( \vec{F} - 2 \vec{a} \).

The equation, \( \vec{r} \cdot (\vec{r} - 2 \vec{a}) = 0 \), tells us that \( \vec{r} \parallel (\vec{r} - 2 \vec{a}) \).

A necessary and sufficient condition for two vectors to be perpendicular is that their scalar or dot product be zero.

Let the point \( O \) lie on the circumference of the circle. \( \vec{OF} \) or \( \vec{r} \) is a variable vector which may move to any point on the circle. If \( \vec{F} \) moves and stops so that point \( P \) lies on point \( O \), we would have no vector. That is

If \( |\vec{F}| = 0 \),
the vector is null and its direction is undefined.

If point \( P \) moves anywhere on the circle except on point \( O \), the vector \( \vec{F} \) forms a right angle with \( \vec{F} - 2 \vec{a} \).

The diameter \( 2 \vec{a} \) divides the circle into a semi-circle. The radius vector \( 2 \vec{a} \) and the variable vector \( \vec{F} \) are drawn from the same origin \( O \).

Hence, the vector that begins from the end of vector \( 2 \vec{a} \) and ends at the terminus of vector \( \vec{F} \) represents the difference vector \( \vec{F} - 2 \vec{a} \).
This proves the theorem:

An angle inscribed in a semi-circle is a right angle.

6. Derive an expression for the area of a square of which

\[ \overline{r} = a_1 \mathbf{i} + a_2 \mathbf{j} \]

is the semi-diagonal.

If the semi-diagonal is \( \overline{r} = a_1 \mathbf{i} + a_2 \mathbf{j} \), then the diagonal is

\[ \overline{m} = 2 (a_1 \mathbf{i} + a_2 \mathbf{j}). \]

By definition:

A square is a quadrilateral whose sides are equal, and whose angles are equal.

Let \( |a| \) represent the length of each side.

Let \( AB = X - \text{Axis} \),

and

\( AD = Y - \text{Axis} \).

The diagonal \( 2 (a_1 \mathbf{i} + a_2 \mathbf{j}) \) is expressed in the components of \( \overrightarrow{a} \) on the \( x-, y- \) axes using the \( \mathbf{i}, \mathbf{j} \) unit vectors respectively.

\[ \overrightarrow{a} \cdot \overrightarrow{a} = \overline{m} = (a_1 \mathbf{i} + a_2 \mathbf{j}) \cdot (a_1 \mathbf{i} + a_2 \mathbf{j}). \]

By combining, we get

\[ \overrightarrow{a} \cdot \overrightarrow{a} = \overline{m} = 2 (a_1 \mathbf{i} + a_2 \mathbf{j}). \]

The figure, triangle \( A \ D \ C \) brings to our mind Pythagorean Theorem:

\[ c^2 = a^2 + b^2. \]

Let us apply this theorem by dotting each side into itself.
Combining the sum of the right-hand member, we get

\[ 2 \overline{a}^2 = 4 (a_2^2 - a_1^2). \]

Dividing by 2, we obtain:

\[ \overline{a}^2 = 2 (a_2^2 - a_1^2). \]

Hence,

The area of the square equals to twice the sum of squares of \( a_1 \) and \( a_2 \).

7. The sum of the squares of the distances of any point \( O \) from the angular points of the triangle exceeds the sum of the squares of its distances from the middle points of the sides by the sum of the squares of half the sides.

![Figure 23](image)

Given triangle \( ABC \) with \( R \) the midpoint of \( AB \), \( R \) the midpoint of \( AC \) and \( Q \) the midpoint of \( BC \). Also given any point \( O \).

To prove:

\[ (OA)^2 + (OB)^2 + (OC)^2 - \sum (CR)^2 + (OP)^2 + (OQ)^2 \neq (CR)^2 + (AP)^2 + (BQ)^2, \]

i.e.
\[(OA)^2 \neq (OB)^2 \neq (OC)^2 = (CR)^2 \neq (OP)^2 \neq (OQ)^2 \neq (CR)^2 \neq (AP)^2 \neq (BQ)^2\].

**Proof:**

Let \(\overline{CA} = \overline{a}\),

and \(\overline{CB} = \overline{b}\).

Then

\[\overline{AB} = \overline{b} - \overline{a}\],
and
\[\overline{BC} = \overline{c} - \overline{b}\],
and
\[\overline{CB} = \overline{b} - \overline{c}\].

Also let
\[\overline{AC} = \overline{c} - \overline{a}\],
and
\[\overline{CA} = \overline{a} - \overline{c}\].

and
\[\overline{OC} = \overline{c} \neq 1/2 (\overline{c} - \overline{a}) = 2\overline{a}/2 - \overline{a} = \overline{a}/2\].

\[\overline{CB} = \overline{c} \neq 1/2 (\overline{b} - \overline{c}) = 2\overline{c}/2 - \overline{c} = \overline{b}/2\].

\[\overline{CA} = \overline{c} \neq 1/2 (\overline{a} - \overline{b}) = 2\overline{a}/2 - \overline{a} = \overline{c}/2\].

\[\overline{AP} = 1/2 (\overline{a} - \overline{c})\].

\[\overline{BQ} = 1/2 (\overline{c} - \overline{b})\].

\[\overline{CR} = 1/2 (\overline{a} - \overline{c})\].

Therefore:

\[(OR)^2 \neq (OQ)^2 \neq (OP)^2 \neq (AP)^2 \neq (BQ)^2 \neq (CR)^2\]

\[= \overline{a}^2/2 \cdot \overline{a}^2/2 \cdot \overline{c}^2/2 \cdot \overline{b}^2/2 \cdot \overline{a}^2/2 \cdot \overline{b}^2/2 \cdot \overline{c}^2/2 \cdot \overline{a}^2/2 \cdot \overline{b}^2/2 \cdot \overline{c}^2/2 \]

\[= 1/4 \overline{a}^2 \neq 2 \overline{a} \overline{c} \neq \overline{c}^2 \neq \overline{b}^2 \neq 2 \overline{b} \overline{c} \neq \overline{c}^2 \neq \overline{a}^2 \neq 2 \overline{a} \overline{b} \neq \overline{b}^2 \neq \overline{c}^2\]

\[= 2 \overline{a} \overline{b} \neq \overline{a}^2 \neq \overline{c}^2 = 2 \overline{c} \overline{b} \neq \overline{b}^2 \neq \overline{a}^2 = 2 \overline{a} \overline{c} \neq \overline{c}^2\]

\[= 1/4 (4 \overline{a}^2 \neq 4 \overline{b}^2 \neq 4 \overline{c}^2)\].
\[ \vec{a}^2 \neq \vec{b}^2 \neq \vec{c}^2 = (\vec{OC})^2 \neq (\vec{OE})^2 \neq (\vec{OC})^2. \]

I conclude that,
\[ (\vec{OR})^2 \neq (\vec{OQ})^2 \neq (\vec{OP})^2 \neq (\vec{AP})^2 \neq (\vec{BQ})^2 \neq (\vec{CR})^2 = (\vec{OA})^2 \neq (\vec{OC})^2 \neq (\vec{OB})^2. \]

Q.E.D.

8. Show that
\[ (\vec{a} - \vec{b}) \times (\vec{a} \neq \vec{b}) = 2 \vec{a} \times \vec{b} \]
and give its geometric interpretation.
\[ (\vec{a} - \vec{b}) \times (\vec{a} \neq \vec{b}) = \vec{a} \times \vec{a} \neq \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}. \]

But
\[ \vec{a} \times \vec{a} = \vec{0}, \]
and
\[ \vec{b} \times \vec{b} = \vec{0}. \]

A necessary and sufficient condition for the cross product to vanish is that their vectors be parallel.

By the definition of the cross product,
\[ \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}. \]

Hence,
\[ (\vec{a} \times \vec{b}) \neq (\vec{a} \times \vec{b}) = 2 \vec{a} \times \vec{b}. \]

Therefore,
\[ (\vec{a} - \vec{b}) \times (\vec{a} \neq \vec{b}) = 2 \vec{a} \times \vec{b}. \]

Or,
\[ \frac{1}{2}(\vec{a} - \vec{b}) \times (\vec{a} \neq \vec{b}) 1 = 2 \frac{1}{2} \vec{a} \times \vec{b}. \]

![Figure 2](image.png)
\[ \mathbf{a} \times \mathbf{b} = \mathbf{E} \; ab \sin (\mathbf{a} \mathbf{b}) = -\mathbf{b} \times \mathbf{a} \]

Given two vectors \( \mathbf{a} \) and \( \mathbf{b} \) the magnitude is the area of the parallelogram \( \mathcal{C} \mathcal{D} \) of which \( \mathbf{a} \) and \( \mathbf{b} \) are the adjacent sides.

\( \mathcal{O} \mathcal{A} \) is a side of the parallelogram \( \mathcal{C} \mathcal{D} \mathcal{E} \); \( \mathcal{C} \mathcal{D} \) is the diagonal of parallelogram \( \mathcal{O} \mathcal{B} \mathcal{D} \mathcal{C} \).

Hence, the parallelogram \( \mathcal{O} \mathcal{B} \mathcal{D} \mathcal{C} \) is two times the area of the parallelogram \( \mathcal{C} \mathcal{D} \mathcal{E} \).

Or,

the area of a parallelogram \( \mathcal{C} \mathcal{D} \mathcal{E} \) whose adjacent sides are \( 1/2 \) of the diagonals of the parallelogram \( \mathcal{O} \mathcal{B} \mathcal{D} \mathcal{C} \), is \( 1/2 \) of the area of the parallelogram \( \mathcal{O} \mathcal{B} \mathcal{D} \mathcal{C} \).

9. Show that

\[ (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a}^2 - \mathbf{b}^2. \]

and interpret.

\[ \frac{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})}{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})} \]

\( (\mathbf{a} - \mathbf{b}) \) and \( (\mathbf{a} + \mathbf{b}) \) represent the difference and the sum of two given vectors \( \mathbf{a} \) and \( \mathbf{b} \) respectively.

The difference and the sum of the two given vectors are the sides of the parallelogram \( \mathcal{O} \mathcal{B} \mathcal{D} \mathcal{C} \).

The dot product of the sum and difference of the two vectors \( (\mathbf{a} - \mathbf{b}) \) and \( (\mathbf{a} + \mathbf{b}) \) equal the difference of the squares of the magnitudes of the
two vectors.

Or, the dot product of the adjacent sides of a parallelogram equals to the difference of the squares of the semi-diagonals of the parallelogram.
