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Some applications of vectors to the study of solid geometry

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SOME APPLICATIONS OF VECTORS TO THE STUDY OF
SOLID GEOMETRY

A THESIS
SUBMITTED TO THE FACULTY OF ATLANTA UNIVERSITY
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BY
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DEPARTMENT OF MATHEMATICS

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CHAPTER I

INTRODUCTION

We differentiate between two types of mathematical quantities; those having magnitude only are called scalars; those having both magnitude and direction are called vectors. In this paper some of the elementary properties of the latter and their application to three dimensional space will be presented.

Vectors arose as a result of the mathematicians' attempts to represent complex numbers geometrically. A first step in this direction was the invention of the quaternion in 1843 by Sir William Hamilton. He made use of four quantities in representing the quaternion: one scalar and the three imaginary units, $i$, $j$, and $k$. The quaternion, $q$ is written in the form $a + bi + cj + dk$. The following laws govern multiplication of the units: $i^2 = j^2 = k^2 = -1$; $ij = -ji = k$; $jk = -kj = i$; $ki = -ik = j$.

Quaternions are easily adapted to matrix algebra. The matrix $\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$ may also be written in the following form:

$$a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$  

If the coefficients of $b$, $c$, and $d$ are called $i$, $j$, and $k$ respectively, we obtain the form $a + bi + cj + dk$. The products
of the units in this expression behave just as those of the quaternions of Hamilton. As an example, we see that $ij = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = k$.

The use of quaternions was not too widespread, for the applied mathematician found them too general and complex. Josiah Willard Gibbs, an American, used some of the ideas introduced by Hamilton and rejected others in developing the field known as Vector Analysis.

This paper is an attempt to show the properties of vectors, apply these properties to problems in three dimensional space, and draw some conclusions from this study.
Definition of a vector.- A vector is defined as a quantity having both magnitude and direction. Associated with every vector are two points— the initial point or head and the terminal point or tail. Graphically, we show the vector PQ as the directed line segment from P to Q.

![Figure 1](image1.png)

**Figure 1**

Vector symbolism.- The vector above may be indicated by the symbol \( \overrightarrow{PQ} \) in which the magnitude is the scalar quantity PQ, and the arrow shows the direction to be from P to Q. In the proofs of theorems, however, this method becomes unwieldy; thus, hereafter a vector will be denoted by a single small letter with a bar to distinguish it from a scalar quantity.

Negative and null vectors.- The set of vectors which have the same magnitude but opposite direction of any given vector is called the negative of the given vector. Thus, in

![Figure 2](image2.png)

**Figure 2**

3.
4.

In the case when the initial point and the terminal point coincide, then we have what is called the zero or null vector. The magnitude of this set of vectors equals zero.

**Equality of vectors.** Equal vectors are defined to be those having the same magnitude and the same direction. This means that all equal vectors are parallel.

**Geometric representation of a vector.** We represent a vector on a Cartesian diagram by plotting the coordinates of its initial point and those of its terminal point. If the coordinates of the initial point and terminal point of \( \vec{a} \) are \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) respectively, then the differences \(x_2 - x_1, y_2 - y_1,\) and \(z_2 - z_1\) are called the components of \( \vec{a} \). Two vectors are equal if and only if their corresponding components are equal.

**Reciprocal vectors.** If \( \vec{a}' \) is parallel to \( \vec{a} \) but \( a' = \frac{1}{a} \), then \( \vec{a}' \) and \( \vec{a} \) are said to be reciprocal vectors.

![Figure 3](attachment:image.png)
CHAPTER III

ALGEBRAIC OPERATIONS

Vector addition. - The algebraic operations of addition, subtraction, and multiplication may be performed on vectors. We define the sum of \( \vec{a} \) and \( \vec{b} \) in the following manner: place the vectors, without changing their directions, so that the terminal point of \( \vec{a} \) coincides with the initial point of \( \vec{b} \); the sum is the vector whose initial point is the head of \( \vec{a} \) and whose terminal point is the tail of \( \vec{b} \).

![Figure 4](image)

By using the definition we can show that the commutative law for vector addition holds, i.e., \( \vec{a} + \vec{b} = \vec{b} + \vec{a} \). The sum of \( \vec{a} \) and \( \vec{b} \) is \( \vec{c} \). By completing the parallelogram in figure 5, we get \( \vec{b} - \vec{a} = \vec{c} \) also; therefore, the law holds.

![Figure 5](image)
6.

The associative law for vector addition states that for any vectors, \( \vec{a} \), \( \vec{b} \), and \( \vec{c} \), \( \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} \). In figure 6, \( (\vec{b} + \vec{c}) = \vec{d} \) and \( \vec{a} + \vec{d} = \vec{e} \). Likewise \( (\vec{a} + \vec{b}) = \vec{f} \) and \( \vec{f} + \vec{c} = \vec{e} \). Thus, the associative law holds.

![Figure 6.](image)

The sum of several vectors is obtained by using the definition repeatedly. This sum is the vector whose head is the initial point of the first addend and whose tail is the terminal point of the last addend.

**Subtraction of vectors.** Subtraction of vectors may be considered as the special case in which we add a vector to the negative of another vector. Then \( \vec{a} - \vec{b} = \vec{a} + (-\vec{b}) \).

Gibbs [1;12] gives a neat geometric interpretation of the subtraction of two vectors. In figure 7, \( OAB'D \) and \( OABC \) are parallelograms. By the definition of addition, \( \vec{a} + \vec{b} = \vec{c} \) and \( \vec{a} + (-\vec{b}) = \vec{d} \). However, \( \vec{d} \) equals and is parallel to \( \vec{e} \); therefore \( \vec{a} - \vec{b} = \vec{e} \).

![Figure 7.](image)
7.

The rule states that the difference between two vectors drawn from the same origin is a vector whose initial point is the tail of the vector representing the subtrahend and whose terminal point is the tail of the vector representing the minuend.

Scalar multiplication.- The product of a scalar quantity, $s$, by $\vec{a}$ is a vector which has been increased $s$-times in magnitude. If $s$ is positive, the direction of the new vector is the same as that of $\vec{a}$; if $s$ is negative, the direction is opposite that of $\vec{a}$. The product $s\vec{a}$ yields the zero vector if $s = 0$. When a vector is multiplied by a scalar, each of its components is multiplied by that scalar.

Both the right and left distributive laws hold for the multiplication of a vector by a scalar, i.e., $(s + s)\vec{a} = \vec{a}(s + s) = s\vec{a} + s\vec{a}$.

Collinear and coplanar vectors.- If there exist scalar quantities, $s$ and $t$ (not both zero), such that $s\vec{a} + t\vec{b} = 0$, then $\vec{a}$ and $\vec{b}$ are called collinear vectors.

In order for three vectors to lie in the same plane they must be linearly dependent. This says scalar quantities, $r$, $s$, and $t$ (not all zero), must exist such that $r\vec{a} + s\vec{b} + t\vec{c} = 0$. Four vectors are always linearly dependent.
Decomposition of vectors. - Coffin \([2; 21]\) shows that any vector may be decomposed into any number of component vectors which may or may not lie in the same plane. A most convenient selection is three mutually perpendicular axes, \(x\), \(y\), and \(z\), along which are placed the unit vectors \(\hat{i}\), \(\hat{j}\), and \(\hat{k}\). A vector, \(\vec{a}\), with components \(a_1\), \(a_2\), \(a_3\), can then be expressed as \(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}\), a quite useful form. The vector is then said to be a linear combination of the unit vectors.
CHAPTER IV

VECTORIAL OPERATIONS

Product of two vectors. - In multiplying one vector by another, one encounters two types of products, scalar and vector products. These products have no analogue in the algebra of real numbers. The scalar product, sometimes called the inner product, is indicated by a raised dot; the vector product, also known as the cross product is indicated by an x.

We define the inner product of two vectors as the product of their lengths multiplied by the cosine of the angle between them. By using the definition we get the following rules for multiplication of the units: \( \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1; \mathbf{i}\mathbf{j} = \mathbf{k} = 0; \mathbf{j}\mathbf{k} = \mathbf{i} = 0; \mathbf{k}\mathbf{i} = \mathbf{j} = 0. \) If two vectors, \( \mathbf{a} \) and \( \mathbf{b} \), are expressed as linear combinations of their components and the unit vectors, we get the scalar product in the following way:

\[
(\mathbf{a} \cdot \mathbf{b}) = (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \cdot (b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}).
\]

Combining terms on the right gives \( a_1b_1 + a_2b_2 + a_3b_3 \). We may conclude that the scalar product of two vectors is the sum of the products of corresponding components.

The cross product of \( \mathbf{a} \) and \( \mathbf{b} \) is the vector whose magnitude is \( ab \sin \theta \) (where \( \theta \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \)) and whose direction is the perpendicular from the plane of \( \mathbf{a} \) and \( \mathbf{b} \) viewed counterclockwise (Figure 8).
Multiplication of the units is given by the following rule: $I^2 = J^2 = K^2 = O_2$; $IJ = -K = E$; $JK = -E = I$; $KE = -E = J$. (Note that $I = I \times I$ etc.)

All of the laws governing multiplication of real numbers do not hold for the latter product. This product is not commutative, but the distributive and the scalar associative laws are valid.

![Figure 8](image)

**Triple products.** It is possible to multiply three vectors and obtain either the triple scalar or the triple vector product.

For any three vectors, $a$, $b$, and $c$, expressions of the type $a(b \times c)$ and $(a \times b)c$ are known as triple scalar products of $a$, $b$, and $c$. This product can be written in terms of the components of the three vectors. By expanding the first expression, we get $(a_1 I + a_2 J + a_3 K) \cdot [(b_1 I + b_2 J + b_3 K) \times (c_1 I + c_2 J + c_3 K)]$. Simplifying the expression in the bracket we obtain $(a_1 I + a_2 J + a_3 K) \cdot [(b_3 c_1 - b_1 c_3)J + (b_1 c_2 - b_2 c_1)K]$. This last form reduces to

$$a \begin{bmatrix} b & c & -b & c \\ 1 & 2 & 3 & 3 & 2 \\ 2 & 3 & 1 & 1 & 3 \\ 3 & 1 & 2 & 2 & 1 \end{bmatrix} \cdot a \begin{bmatrix} b & c & -b & c \\ 1 & 2 & 3 & 3 & 2 \\ 2 & 3 & 1 & 1 & 3 \\ 3 & 1 & 2 & 2 & 1 \end{bmatrix}.$$
For any three vectors \( \vec{a}, \vec{b}, \) and \( \vec{c} \), an expression of the form \( \vec{a} \times (\vec{b} \times \vec{c}) \) is called a triple vector product. This product could also be expanded in terms of its components to give the form 
\[
(a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) \cdot (1 1 1 - 2 2 3 - 3 3 1) = 0
\]
More concisely, this is written in the form
\[
(a \cdot \vec{b}) \cdot (\vec{a} \cdot \vec{c}) - (a \cdot \vec{b}) \cdot (\vec{a} \cdot \vec{c})
\]

**Vector product of four vectors.** - The importance of the triple scalar product lies in the fact that all higher products can be reduced to the triple product. Consider the vector product \((\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})\). By making use of a reduction formula we may write the following two expressions:
\[
\vec{c}(\vec{a} \vec{b} \vec{d}) - \vec{d}(\vec{a} \vec{b} \vec{c}) \quad \text{and} \quad \vec{b}(\vec{a} \vec{c} \vec{d}) - \vec{d}(\vec{a} \vec{c} \vec{b})
\]
When combined, these two expressions give
\[
\vec{a}(\vec{b} \vec{c} \vec{d}) - \vec{b}(\vec{a} \vec{c} \vec{d}) + \vec{c}(\vec{a} \vec{b} \vec{d}) - \vec{d}(\vec{a} \vec{b} \vec{c}) = 0,
\]
which is the condition for four vectors to be linearly dependent.
A few problems involving well-known properties of the sphere and tetrahedron have been solved as a means of showing how vectors are applied to Solid Geometry.

**Problem I:** Express the equation of a sphere in vectorial form.- A sphere is completely determined if its radius and the position of its center are known. In the sphere, S, (Figure 9) suppose P is a point on the sphere and C is the center. Let the coordinates of P be \( (p_1, p_2, p_3) \) and those of C be \( (c_1, c_2, c_3) \). We may then write
\[
|\overline{CP}|^2 = |\vec{a}|^2 = \sum_{i=1}^{3} (p_i - c_i)^2 = (p_1 - c_1)^2 - (p_2 - c_2)^2 - (p_3 - c_3)^2,
\]
or
\[
|\vec{a}|^2 = (p_1 - c_1)^2 + (p_2 - c_2)^2 + (p_3 - c_3)^2.
\]
This is the equation of the sphere in terms of vectors.

![Figure 9](image-url)
13.

**Problem II:** Show that the joins of the midpoints of opposite edges of a tetrahedron bisect each other. - Let A, B, C, and D be the vertices of a tetrahedron, and let \( \vec{a}, \vec{b}, \vec{c}, \) and \( \vec{d} \) be the respective position-vectors of the vertices with respect to an arbitrary origin, \( O \). Let \( I, \vec{m}, \vec{n}, \vec{u}, \vec{v}, \) and \( \vec{w} \) represent the position-vectors of the midpoints of the edges with respect to \( O \) (Figure 10). Adding vectors \( \vec{a} \) and \( \vec{b} \) gives the sum \( AB \) since the diagonals of a parallelogram are equal. However, \( AB = 2\vec{n} \); therefore \( \vec{n} = \frac{\vec{a} + \vec{b}}{2} \). In a similar way, we obtain these additional equations: \( I = \frac{\vec{b} + \vec{c}}{2}, \vec{m} = \frac{\vec{c} + \vec{a}}{2}, \vec{u} = \frac{\vec{a} + \vec{d}}{2}, \vec{v} = \frac{\vec{b} + \vec{d}}{2}, \) and \( \vec{w} = \frac{\vec{c} + \vec{d}}{2} \). The lines which we want to bisect each other are \( IU, NW, \) and \( MV \). The midpoints of these lines are the points of intersection of the diagonals of parallelograms \( OLUX, ONX, \) and \( OMNX \) respectively, where \( X, X, \) and \( X \) represent the unknown vertices of the completed parallelograms. It is now necessary to show that these midpoints are coincident. The position-vectors of these points are \( \frac{I + U}{2}, \frac{M + V}{2}, \) and \( \frac{N + W}{2} \), each of which is equal to the expression \( \frac{\vec{a} + \vec{b} + \vec{c} + \vec{d}}{4} \). Therefore, the lines bisect each other.
Figure 10.
Problem III. Show that the expression \( \frac{PA'}{AA'} + \frac{PB'}{BB'} + \frac{PC'}{CC'} + \frac{PD'}{DD'} \) holds if \( P \) is a point within tetrahedron \( ABCD \) and \( A', B', C', \) and \( D' \) are the points where the lines joining \( P \) to each vertex of the tetrahedron intersect the opposite face.

The four vectors involved may be reduced to three by choosing an arbitrary vertex, say \( A \), as the origin, and then writing \( \overrightarrow{b} = \overrightarrow{AB}, \overrightarrow{c} = \overrightarrow{AC}, \) and \( \overrightarrow{d} = \overrightarrow{AD} \) (Figure 11). \( \overrightarrow{AP} = \overrightarrow{p} \) can be written as a linear combination of \( \overrightarrow{b}, \overrightarrow{c}, \) and \( \overrightarrow{d} \); then \( \overrightarrow{p} = r\overrightarrow{b} + s\overrightarrow{c} + t\overrightarrow{d} \), where \( r, s, \) and \( t \) are not all zero. Since \( AA' = \overrightarrow{a}' \) is a scalar multiple of \( \overrightarrow{p} \), \( \overrightarrow{a}' = k(r\overrightarrow{b} + s\overrightarrow{c} + t\overrightarrow{d}) \); also \( \overrightarrow{a}' = \overrightarrow{b} + \overrightarrow{BA}' \). Since \( \overrightarrow{BA}' \) is coplanar with \( (\overrightarrow{c} - \overrightarrow{b}) \) and \( (\overrightarrow{d} - \overrightarrow{b}) \), it may be expressed in terms of them. Then \( \overrightarrow{a}' = m(\overrightarrow{c} - \overrightarrow{b}) + n(\overrightarrow{d} - \overrightarrow{b}) + \overrightarrow{b} \). By equating the coefficients of \( \overrightarrow{b}, \overrightarrow{c}, \) and \( \overrightarrow{d} \) from the two expressions of \( \overrightarrow{a}' \), we obtain \( k = \frac{1 - m - n}{r}, k = \frac{m}{s}, \) and \( k = \frac{n}{t} \)

which when combined give \( k = \frac{1}{r + s + t} \). From the figure we see that \( \overrightarrow{PA}' = \overrightarrow{AA}' - \overrightarrow{AP} \). Substituting the values of \( \overrightarrow{AA}' \) and \( \overrightarrow{AP} \) and simplifying, we find that \( \overrightarrow{PA}' = (k - 1)(r\overrightarrow{b} + s\overrightarrow{c} + t\overrightarrow{d}) \). Then \( \frac{PA'}{AA'} = k - 1 - 1 - (r + s + t) \). Using the same method we find that \( \frac{PB'}{BB'} = r, \frac{PC'}{CC'} = s, \) and \( \frac{PD'}{DD'} = t \). Addition of these four values gives the required results.
CHAPTER VI

CONCLUSIONS

Since quantities such as force, velocity, acceleration, etc. have direction and magnitude associated with them, the applied mathematician will find vectors to be a most desirable aid. The pure mathematician is able to revel in the fact that the conversion of theorems from conventional form to vectorial form is possible; but this conversion, in many cases, is not a simplification.

Both the usefulness and beauty of vectors, when considered together, serve to make them an integral part of the field of mathematics.
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