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On the axiom of choice

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ON THE AXIOM OF CHOICE

A THESIS

SUBMITTED TO THE FACULTY OF ATLANTA UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF MASTER OF SCIENCE

BY
EMMA ELsie SCHELL

DEPARTMENT OF MATHEMATICS

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\[ R = 6 \quad T = 9 \]
### SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$S = \sum_{A \in Z} A$</td>
<td>$S$ equals the sum of $A$ where $A$ belongs to $Z$.</td>
</tr>
<tr>
<td>$B \subseteq S$</td>
<td>Each point of $B$ belongs to $S$; $B$ is a subset of $S$.</td>
</tr>
<tr>
<td>$Z = {A}$</td>
<td>The set $Z$ consists of a single element $A$.</td>
</tr>
<tr>
<td>$Z \sim B$</td>
<td>The sets $Z$ and $B$ can be put into one-to-one correspondence.</td>
</tr>
<tr>
<td>$A \in Z$</td>
<td>The point $A$ belongs to the set $Z$.</td>
</tr>
<tr>
<td>$&lt;$</td>
<td>Less than</td>
</tr>
<tr>
<td>$&gt;$</td>
<td>Greater than</td>
</tr>
<tr>
<td>$\leq$</td>
<td>Less than or equal to</td>
</tr>
<tr>
<td>$\geq$</td>
<td>Greater than or equal to</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>Function of $x$</td>
</tr>
<tr>
<td>$a'$</td>
<td>A prime</td>
</tr>
<tr>
<td>$X \cup Y$</td>
<td>The union of $X$ and $Y$</td>
</tr>
<tr>
<td>$X \cap Y$</td>
<td>The intersection of $X$ and $Y$</td>
</tr>
<tr>
<td>$\bar{A} &lt; \bar{B}$</td>
<td>The set $A$ is of less power than the set $B$.</td>
</tr>
<tr>
<td>$\bar{A}$</td>
<td>The order type of $a$.</td>
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CHAPTER I

INTRODUCTION

Our intention is to discuss the axiom of choice. We shall give some applications of the axiom of choice and show the equivalence of the axiom to the well-ordering theorem.

Historically, the axiom of choice was first introduced by Zermelo (1904) in order to prove that every set can be well-ordered. Until the last two decades, probably the main application of the axiom in general mathematics was through the well-ordering theorem and its applications. The introduction of the axiom of choice gave rise to a lively exchange of opinion among mathematicians. Today we have an extensive bibliography on the subject of this axiom and its applications. A statement of the axiom follows.

Axiom of Choice.—For every set $Z$ whose elements are sets $A$, non-empty, having no common elements, there exists at least one set $P$ having one and only one element from each of the sets $A$ belonging to $Z$.

Other wordings of the axiom of choice are as follows:

If a non-empty set $S$ is the sum of the disjoint non-empty sets, then there exists at least one subset of $S$ which has one and only one element with each of those sets.

1Abraham A. Fraenkel, Foundations of Set Theory, Amsterdam, 1958, p. 80.
For any set $A$ there is a function $f$ such that for any non-empty subset $B$ of $A$, $f(B) \in A$.

Given any collection of sets, there exists a "method" of designating a particular element of each set as a "special" element of that set; for any collection $A$ of sets there exists a single-valued function $f$ such that $f(S)$ is an element of $S$ for each set $S$ of the collection $A$.

The simplest case of the axiom of choice is that which the set $Z$ consists of a single set $A$, $Z = \{ A \}$. Then, of course the axiom of choice is reduced to the statement that if the set $A$ is not empty, then there exists at least one object forming an element of the set $A$. This statement, however, is true, since the propositions "the set $A$ is not empty" and "there exists at least one object forming an element of the set $A"$ are equivalent.

**Definitions:**

**Denumerable set** - A set which is equivalent to the set of all natural numbers is called a denumerable set.

**Disjoint sets** - Two sets are disjoint if there is no point which belongs to each of the sets.

**Empty set** - A set is empty if it has no members.

**Equivalent sets** - Sets are equivalent if they can be put into one-to-one correspondence.

**One-to-one correspondence** - A function that always maps distinct elements onto distinct elements is called one-to-one (usually a one-to-one correspondence).

**Power of the continuum** - A set which is equivalent to
the set of all real numbers is said to be of the power of the continuum.

Sets of less and greater power - Let A and B be two given sets. If the set A is equivalent to (i.e., of the same power as) a certain subset of the set B, but the set A, then we shall say that the set A is of less power than the set B and write $A < B$.

Well-ordered sets - An ordered set is said to be well-ordered if each non-empty subset of that set has a first element.

Ordinal number - An ordinal number is defined as the order type of a well-ordered set.

Order type - The order type of a set is that property of the set which remains when we disregard the quality of the elements of that set but not their order.
CHAPTER II

GENERAL DEVELOPMENT

We shall now pass to some applications of the axiom of choice. As early as 1902, i.e., a few years before Zermelo announced the axiom of choice, Beppo Levi pointed out that in the general case we are not able to prove that the sum $S$ of disjoint non-empty sets forming a set $Z$ of sets is of greater power than, or of the same power as, the set $Z$ and that the proof can be given in those cases in which we are able to distinguish one element in each of the sets forming the set $Z$.

Referring to the axiom of choice, we can prove the general case as follows: By virtue of the axiom of choice there exists a set $B$ having one and only one element from each of the sets $A$ forming the set $A$. Since

$$S = \sum_{A \in Z} A,$$

we have $B \subseteq S$. On the other hand, we have $Z \sim B$; in order to obtain a (1-1) correspondence between the elements of the sets $A$ and $B$ it suffices to associate with every set $A \in Z$ the only element of the set $A \cap B$. Therefore, the set $Z$ is equivalent to a certain subset $B$ of the set $S$, whence, by virtue of the axiom of choice for finite sets, we find $\overline{Z} < \overline{S}$.

Thus we have proved, with the aid of the axiom of choice

Theorem 1. If we decompose any set of $A$ into disjoint non-empty subsets, then the set of all those subsets is of power $\leq$ the power of the set $A$. 
In his classical paper of 1904, Zermelo was concerned to show that the axiom of choice implies that every set can be well-ordered. In that paper he used the following formulation:

(1) If A is a set of non-empty, pairwise disjoint sets, then there is a set C whose intersection with any number B has exactly one element, that is, \( C \cap B \) is a unit set.

Our objectives at this point are to state a number of principles equivalent to the axiom of choice and to prove that the axiom of choice implies the well-ordering theorem. We begin with:

**Theorem 2.** (1) is equivalent to the axiom of choice.

Another common formulation is:

(2) Given any relation R there is a function \( f \in R \) such that the domain of R = domain of f.

**Theorem 3.** (2) is equivalent to the axiom of choice.

Our program is now to establish the following equivalence:

The axiom of choice is equivalent to well-ordering theorem (every set can be well-ordered). Before showing this equivalence, let us prove the following:

**Theorem 4.** For every set there exists a correspondence according to every non-empty subset of that set corresponds a certain element of that subset.

We shall prove that theorem 4 is equivalent to the axiom of choice. Therefore, let us assume that the axiom of choice is true and let M denote an arbitrary set. For every non-empty subset \( N \) of the set M let us denote by \( A_N \)
the set of all ordered pairs \((P,N)\) where \(P \in N\), and by \(Z\) - the set of all the sets \(A_N\) where \(0 \neq N \subset M\). The sets \(A_N\) forming the set \(Z\) are, of course, non-empty and have no common elements. Therefore, by the axiom of choice, there exists a set \(B\) having one and only one element from each of the sets belonging to \(Z\). Thus for every set \(N\) such that \(0 \neq N \subset M\) the set \(A_N \cap B\) consists of only one element, which we shall denote by \(P_N \cap N\). For \(0 \neq N \subset M\), let \(\alpha(N) = P_N\). Clearly the function \(\alpha\) associates with each non-empty subset \(N\) of the set \(M\) a certain element \(\alpha(N)\) of that subset. Therefore, theorem 4 is true.

Thus we have proved that theorem 4 follows from the axiom of choice.

In order to prove that the axiom of choice follows from theorem 4, let us assume that theorem 4 is true, and let \(Z\) denote any set whose elements are non-empty sets \(A\) having no common elements. Let \(M\) denote the sum of all the sets \(A\) forming the set \(Z\). By theorem 4, there exists a function \(\alpha\) such that for \(0 \neq N \subset M\) we have \(\alpha(N) \in N\). Since for \(A \in Z\) we have \(0 \neq A \subset M\), we shall have \(\alpha(A) \in A\) for \(A \in Z\).

Let us denote by \(B\) the set of all elements \(\alpha(A)\) where \(A \in Z\). Thus we shall have \(\alpha(A) \in A \cap B\) for \(A \in Z\), and since the sets \(A\) forming the set \(Z\) have no common elements, \(\alpha(A)\) is the only element of the set \(A \cap B\). Thus the set \(B\) has one and only one element from each of the sets \(A\) belonging to \(Z\). Therefore, the axiom of choice is true.
Thus we have proved the equivalence of the axiom of choice and theorem 4.

Theorem 5. Knowing the function $f$ associating with each non-empty subset, $x$ of a given set $M$ a certain element of that subset, $f(x) \in x$, we are able to define the well-ordering of the set $M$.

Proof. Let $M$ denote a given set of power $m$, $\mathbb{Z}$- the set of all ordinal numbers $\alpha$ such that $\alpha \in m$, $\mathcal{O}$- the order type of the set $\mathbb{Z}$ ordered according to the magnitude of the numbers belonging to it.

Let $\alpha = f(M)$. Now let $\alpha$ denote an ordinal number $> 0$ and suppose that we have already defined all elements $\alpha_\beta$, where $\beta < \alpha$ as certain elements of the set $M$; let $M_\alpha$ denote their set. If we had $M_\alpha \neq M$, then the set $M - M_\alpha$ would be non-empty and we could set $\alpha_\alpha = f(M - M_\alpha)$. The elements $\alpha_\alpha$ are thus defined by transfinite induction for every ordinal number $\alpha > 0$ for which the set $M_\alpha$ is different from the set $M$. We shall prove that this cannot hold for every ordinal number $\alpha < \mathcal{O}$. Indeed, in that case, we could define by transfinite induction all elements $\alpha_\gamma$ for $\gamma < \mathcal{O}$. Those elements are all different from one another because $\alpha_\alpha = f(M - M_\alpha) \in M - M_\alpha$. Therefore, $\alpha_\alpha \neq M_\alpha = \{\alpha_\gamma \mid \gamma < \mathcal{O}\}$ and thus $\alpha_\alpha \neq \alpha$ for $\gamma < \mathcal{O}$. The set $M = \{\alpha_\gamma \mid \gamma < \mathcal{O}\}$ would thus be of power $\mathcal{O}$ and therefore it would not be of power $\leq m$.

Hence, we have a contradiction since all its elements belong to the set $M$ of power $m$. We have proved that there exists an ordinal number, only one, $\alpha < \mathcal{O}$ such that $M_\alpha = M$. Then $M = \{\alpha_\gamma \mid \gamma < \mathcal{O}\}$ and the set $M$ is well-ordered of type $\alpha$. 

Theorem 5 is thus proved.

Theorem 6 immediately follows.

**Theorem 6.** It follows from the axiom of choice that for every set there exists a relation $Q$ well-ordering that set.

**Proof.** We have proved that from the axiom of choice follows theorem 4 stating that for every set $M$ there exists a function $f$ associating with each non-empty subset $X$ of the set $M$ a certain element of that subset. Hence and from theorem 5 immediately follows the validity of theorem 6.

The assertion that for every set there exists a relation well-ordering that set is called the well-ordering theorem of Zermelo. By virtue of theorem 6, Zermelo's theorem results from the axiom of choice. Now to prove that conversely, the axiom of choice, results from Zermelo's theorem.

Let $Z$ denote a set whose elements are non-empty sets having no common elements. Let $S$ be the sum of all the sets $A$ forming the set $Z$ of sets. By virtue of Zermelo's theorem, there exists a relation $Q$ well-ordering the set $S$. Each set $A$ belonging to $Z$ is, of course, a non-empty subset of the well-ordered set $S$; let $f(A)$ denote the first of the elements of the set $S$ belonging to $A$. The set $B$ of all elements of $f(A)$, where $A \in Z$, will, of course, contain one and only one element of each of the sets $A$ belonging to $Z$.

The axiom of choice is thus true.

Hence the axiom of choice is equivalent to the well-ordering theorem.
BIBLIOGRAPHY


