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On infinite products and infinite integrals

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ON INFINITE PRODUCTS AND INFINITE INTEGRALS

A THESIS

SUBMITTED TO THE FACULTY OF ATLANTA UNIVERSITY
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FOR THE DEGREE OF MASTER OF SCIENCE

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CHAPTER I

INTRODUCTION

This study concerns itself with convergence and uniform convergence of infinite products and infinite integrals. However, since many of the definitions and theorems commonly met in discussion of convergence and uniform convergence for infinite products and infinite integrals have their more elementary counterparts in discussions of convergence and uniform convergence for infinite sequences and infinite series, the author found it convenient to start with a discussion of the latter two.

No new theorems are presented herein, rather, an attempt is made to set things down in such a way that analogies between definitions, theorems and proofs of the different sections can easily be seen and can be used for clarification.

An attempt is made to prove all theorems stated, except in cases where the proof is immediate from a preceding theorem and/or an analogous proof has been given in an earlier section. One exception is made to the above in chapter seven where the fundamental theorem of integral calculus is stated without proof.

Examples are given to illustrate the meanings and implications of certain important definitions and theorems.
Definitions are given and notations are explained as they are needed.
CHAPTER II

CONVERGENCE AND UNIFORM CONVERGENCE
OF A SEQUENCE OF FUNCTIONS

**Definition 2.1.** Let \( \{ f_n(x) \} = f_1(x), f_2(x), \ldots, f_n(x), \ldots \) be a sequence of functions. \( \{ f_n(x) \} \) is said to converge at a point \( x_0 \) if the sequence of numbers \( \{ f_n(x_0) \} \) is convergent.

**Definition 2.2.** Let \( \{ f_n(x) \} \) be defined on a set \( A \). We say the sequence converges on \( A \) in case, for every fixed \( x \) in \( A \), the sequence of constants \( \{ f_n(x) \} \) converges.

**Definition 2.3.** The sequence \( \{ f_n(x) \} \) is said to converge uniformly on a set \( A \) if for every positive real number \( \varepsilon \) there exists a positive integer \( N \) depending only on \( \varepsilon \) such that for every \( m, n > N \) and for every \( x \) in \( A \),

\[
| f_m(x) - f_n(x) | < \varepsilon.
\]

**Note 2.1.** If \( \{ f_n(x) \} \) converges according to definition 2.1 we write

\[
\lim_{n \to \infty} f(x_0) = f(x_0).
\]

If \( \{ f_n(x) \} \) converges according to definition 2.3 we write

\[
\lim_{n \to \infty} f_n(x) = f(x).
\]

**Definition 2.4.** (Negation of uniform convergence). A
sequence of functions \( \{ f_n(x) \} \) defined on a set \( A \), fails to converge uniformly on \( A \) to a function \( f(x) \) if and only if there exists a positive number \( \varepsilon \) having the property that for any number \( N \) there exists a positive integer \( n > N \) and a point \( x \) in \( A \) such that

\[
\| S_n(x) - S(x) \| \geq \varepsilon.
\]

Convergence and uniform convergence are far from being the same thing. As a matter of fact uniform convergence implies convergence, but not conversely. Examples which illustrate the difference are as follows:

**Example 2.1.** Let \( A \) be the set of all real numbers and \( \{ f_n(x) \} = \{ \frac{x}{n} \} \) a sequence of functions defined on \( A \).

(a) \( \{ f_n(x) \} \) converges on \( A \). For suppose \( x_0 \) belongs to \( A \). Let \( \varepsilon > 0 \) and \( N > \frac{2x_0}{\varepsilon} \). For every \( m, n \in \mathbb{N} \),

\[
\left| f_n(x_0) - f_m(x_0) \right| = \left| \frac{x_0}{n} - \frac{x_0}{m} \right|
\leq \left| \frac{x_0}{n} \right| + \left| \frac{x_0}{m} \right|
< \left| \frac{x_0}{N} \right| + \left| \frac{x_0}{N} \right|
= \frac{2|x_0|}{N} < \varepsilon
\]

Hence \( \{ f_n(x_0) \} \) converges for every \( x_0 \) in \( A \), so that \( \{ f_n(x) \} \) converges on \( A \).

(b) \( \{ f_n(x) \} \) does not converge uniformly on the set \( A \).

For let \( N \) be any positive integer. We show that there are an \( x_0 \) in \( A \) and integers \( m, n > N \) such that
Indeed if $x_0 = 6N$, $n = 2N$, and $m = 3N$, then

$$|f_n(x_0) - f_m(x_0)| = \left| \frac{6N}{2N} - \frac{5N}{3N} \right| = 1.$$  

This shows that $\{f_n(x)\}$ does not converge uniformly on $A$.

**Example 2.2.** Let $A$ be the set of all real numbers and

$$\{f_n(x)\} = \left\{ \frac{1}{1 + nx^2} \right\} \text{ defined on } A.$$

(a) $\{f_n(x)\}$ converges on $A$. For let $x_0$ belong to $A$.

Let $\epsilon > 0$ and $N > \frac{2}{x_0^2}$. For every $n, m > N$

$$\frac{1}{1 + nx_0^2} - \frac{1}{1 + mx_0^2} = \frac{(n - m)x_0^2}{1 + mnx_0^2 + nx_0^2 + mx_0^2} < \frac{|n - m|x_0^2}{mnx_0^2} < \frac{2}{N|x_0^2} < \epsilon.$$  

Hence, the sequence $\{f_n(x_0)\}$ converges for every $x_0$ in $A$, so that it converges on $A$.

(b) The convergence of $\{f_n(x)\}$ on $A$ is not uniform.

For let $\epsilon = \frac{1}{12}$ and $N$ be arbitrary. If $x_0 = \frac{1}{\sqrt{N}}$, $n = 2N$, and $m = 3N$, then

$$\frac{1}{1 + nx_0^2} - \frac{1}{1 + mx_0^2} = \frac{N^2x_0^2}{(1 + 2Nx_0^2)(1 + 3Nx_0^2)} = \frac{1}{12}.$$  

Hence $\{f_n(x)\}$ does not converge uniformly on $A$. 


(c) However, the convergence is uniform on any closed interval which does not contain zero. Let \([a, b]\) be any closed interval with \(a > 0\). Let \(\epsilon > 0\) and let \(N > \frac{2}{\epsilon a^2}\). Then, for every \(x\) in \([a, b]\) and \(m, n > N\)

\[
|f_n(x) - f_m(x)| < \left|\frac{(n - m)x^2}{mnx^4}\right| < \frac{2}{nx^2} < \frac{2}{Na^2} < \epsilon.
\]

Hence the convergence of \(\{f_n(x)\}\) on \([a, b]\) is uniform.

**Example 2.3.** The sequence \(\frac{x^e}{e^{nx}}\) converges uniformly on \([0, +\infty)\). For observe that given a fixed value of \(n\), the non-negative function \(f(x) = xe^{-nx}\) can be shown (by differentiating) to have a maximum on \([0, +\infty)\) given by \(x = \frac{1}{n}\) and equal to \(\frac{1}{ne}\). This maximum does not depend on the value of \(x\) in \([0, +\infty)\). Hence the function converges uniformly to 0 on \([0, +\infty)\).

We have shown by examples that convergence does not necessarily imply uniform convergence. To show that uniform convergence always implies convergence we offer the following theorems:

**Theorem 2.1.** Let \(\{f_n(x)\}\) be uniformly convergent on a set \(A\), then \(\{f_n(x)\}\) converges on \(A\).

**Proof:** The proof is trivial. By note 2.1 uniform convergence of \(\{f_n(x)\}\) on \(A\) means that for any \(\epsilon > 0\) we can find \(N\) such that \(n > N\) implies \(|f_n(x) - f_m(x)| < \epsilon\) for every \(x\) in \(A\).

If this holds for every \(x\) in \(A\) it certainly holds for a particular \(x\) say \(x_0\). Q. E. D.

The example \(\{f_n(x)\} = \left\{x^{\frac{1}{2n-1}}\right\}, -1 \leq x \leq 1\) shows that
the limit of a sequence of continuous functions need not be continuous. The limit in this case is the signum function, 
\[ f(x) = \begin{cases} 
 1 & \text{if } x > 0, \\
 0 & \text{if } x < 0, \\
 0 & \text{if } x = 0, 
\end{cases} \]
which is discontinuous at \( x = 0 \). We shall see that this is possible because the convergence is not uniform.

In case of uniform convergence we have the basic theorem:

**Theorem 2.2.** If \( \{f_n(x)\} \) is a sequence of functions defined on a set \( A \) and uniformly convergent on \( A \), then if each \( f_n(x) \) is continuous at a point \( a \) belonging to \( A \), it follows that 
\[ f(x) = \lim_{n \to +\infty} f_n(x) \]
is continuous at \( a \).

**Proof:** Let \( \varepsilon > 0 \). Since \( \{f_n(x)\} \) is uniformly convergent on \( A \), there is an \( N \) such that for \( n > N, \ |f(x) - f_n(x)| < \frac{\varepsilon}{3} \) for every \( x \) in \( A \). Let \( n > N \). Now since \( f_n(x) \) is continuous at \( x = a \), there is a \( \delta > 0 \) such that if \( |x - a| < \delta \) where \( x \) belongs to \( A \) then \( |f_n(x) - f_n(a)| < \frac{\varepsilon}{3} \). Now suppose \( |x - a| < \delta \) and \( x \) belongs to \( A \). Then
\[
\left| f(x) - f(a) \right| \leq \left| f(x) - f_n(x) \right| + \left| f_n(x) - f_n(a) \right| + \left| f_n(a) - f(a) \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .
\]
This shows that \( f(x) \) is continuous at \( x = a \).

It is now clear that continuity is not necessarily preserved by ordinary convergence but is preserved by uniform convergence.
CHAPTER III

CONVERGENCE AND UNIFORM CONVERGENCE

OF A SERIES OF FUNCTIONS

**Definition 3.1.** Let \( f_1(x) + \ldots + f_n(x) + \ldots = \sum_{n=1}^{\infty} f_n(x) \)

\( \sum f_n(x) \) be a series of functions defined on a set \( A \), and let \( S_n(x) = f_1(x) + f_2(x) + \ldots + f_n(x) \). We say that this series of functions converges on \( A \) in case the sequence \( \{S_n(x)\} \) converges on \( A \).

**Definition 3.2.** The series \( f_1(x) + f_2(x) + \ldots \) converges uniformly on \( A \) if and only if, the sequence \( \{S_n(x)\} \) converges uniformly on \( A \).

Corresponding to the condition \( a_n \to 0 \) which is necessary for the convergence of the series of constants \( \sum a_n \), we have:

**Theorem 3.1.** (A necessary condition for uniform convergence)

If the series \( \sum f_n(x) \) converges uniformly on a set \( A \), then the general term \( f_n(x) \) converges to 0 uniformly.

**Proof:** By the triangle inequality, if \( S_n(x) = f_1(x) + \ldots + f_n(x) \) and \( S(x) = \sum_{n=1}^{\infty} f_n(x) \),

\[
|f_n(x)| = |S_n(x) - S_{n-1}(x)| = \left| \left[ S_n(x) - S(x) \right] + [S(x) - S_{n-1}(x)] \right| \\
\leq |S_n(x) - S(x)| + |S_{n-1}(x) - S(x)|.
\]
Let \( \epsilon > 0 \) be given. If \( N \) is chosen such that \( n > N - 1 \) implies
\[
| S_n(x) - S(x) | < \frac{\epsilon}{2}
\]
for all \( x \) in \( A \), then \( n > N \) implies \( | f_n(x) | < \epsilon \) for all \( x \) in \( A \).

**Example 3.1.** Want to show that the Maclaurin series for \( e^x \) converges uniformly on a set \( A \), if and only if, \( A \) is bounded.

**Solution:** If the set \( A \) is bounded, it is contained in some interval of the form \( [-\alpha, \alpha] \). Using the Lagrange form of the remainder in Taylor's formula for \( e^x \) at \( x = a = 0 \), we have (with the standard notation), for any \( x \),
\[
| S_n(x) - S(x) | = | R_n(x) | = \frac{e^{\alpha}}{n!} | x |^n \leq \frac{e^\alpha}{n!} \alpha^n.
\]
Since \( \lim_{n \to \infty} \frac{e^\alpha}{n!} \alpha^n = 0 \) and \( \frac{e^\alpha}{n!} \alpha^n \) is independent of \( x \), the uniform convergence on \( A \) is established.

If the set is unbounded, we can show that \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) fails to converge uniformly on \( A \) by showing that the general term does not approach 0 uniformly on \( A \). This we do with the aid of definition 2.4. Letting \( \epsilon = 1 \) and \( n \) be any fixed positive integer, we can find an \( x \) in \( A \) such that \( |x|^n \) is not bounded.

**Theorem 3.2.** (A necessary and sufficient condition for uniform convergence). The series \( \sum f_n(x) \) is uniformly convergent on a set \( A \), if and only if, the following is satisfied: Given \( \epsilon > 0 \), we can find \( N \), depending on \( \epsilon \), but not on \( x \), such that \( | S_m(x) - S_n(x) | < \epsilon \) whenever \( n, m > N \) and for every \( x \) in \( A \).

**Proof:** The condition is easily seen to be necessary. This
follows from definition 2.3.

Now suppose the condition is satisfied. Then by the theorem for ordinary convergence the series $\sum_{n} f_n(x)$ is convergent for every $x$. Let its sum be $S(x)$. Given $\epsilon > 0$, choose $N$ so that

$$|S_m(x) - S_n(x)| < \epsilon \quad (n, m > N)$$

keeping $m$ fixed, make $n \to +\infty$. Then since $S_n(x) \to S(x)$,

$$|S_m(x) - S(x)| < \epsilon$$

provided only that $m > N$. Hence the convergence is uniform.

*By ordinary convergence we mean convergence of a series of constant terms. We assume here all the theorems for convergence of a sequence of constants.*
CHAPTER IV

TESTS FOR UNIFORM CONVERGENCE
OF A SERIES OF FUNCTIONS

Definition 4.1. (Dominance). The statement that a series of functions \( \sum g_n(x) \) dominates a series of functions \( \sum f_n(x) \) on a set \( A \) means that all terms are defined on \( A \) and that for any \( x \) in \( A \) \( |f_n(x)| \leq g_n(x) \) for every positive integer \( n \).

Theorem 4.1. (Comparison test). Any series of functions \( \sum f_n(x) \) dominated on a set \( A \) by a series of functions \( \sum g_n(x) \) which is uniformly convergent on \( A \) is uniformly convergent on \( A \).

Proof: By ordinary convergence we know that the series \( \sum f_n(x) \) converges for every \( x \) in \( A \). If \( f(x) = \sum f_n(x) \) and \( g(x) = \sum g_n(x) \), we have

\[
\begin{align*}
|f_1(x) + \ldots + f_n(x)| - f(x) &= |f_{n+1}(x) + f_{n+2}(x) + \ldots| \\
&\leq |g_{n+1}(x) + g_{n+2}(x) + \ldots| \\
&= |[g_1(x) + \ldots + g_n(x)] - g(x)|.
\end{align*}
\]

If \( \epsilon > 0 \) is given and if \( N \) is chosen such that \( n > N \) implies

\[
|[g_1(x) + \ldots + g_n(x)] - g(x)| < \epsilon
\]

for every \( x \) in \( A \), then

\[
|f_1(x) + \ldots + f_n(x)| - f(x) < \epsilon
\]

for every \( x \) in \( A \).
Theorem 4.2. (Weierstrass' M-test). Let \( \{M_n\} \) be a sequence of nonnegative numbers such that
\[
0 \leq |f_n(x)| \leq M_n, \quad \text{for } n = 1, 2, 3, \ldots \text{ and}
\]
for every \( x \) in a given set \( A \). Then if \( \sum M_n \) converges \( \sum f_n(x) \) converges uniformly on \( A \).

**Proof:** The proof is exactly analogous with the proof of theorem 4.1.

**Example 4.1.** Want to show that the series \( 1 + e^{-x}\cos x + e^{-2x}\cos 2x + \ldots \) converges uniformly on any set that is bounded below by a positive constant.

**Solution:** If \( a > 0 \) is a lower bound for the set \( A \), then for any \( x \) in \( A \), \( |e^{-n}\cos nx| \leq e^{-nx} \). By the Weierstrass M-test, with \( M_n = e^{-na} \), the given series converges uniformly on \( A \) (the series \( \sum e^{-na} \) is a geometric series with common ratio \( e^{-a} < 1 \)).

**Example 4.2.** Observe that the series of functions \( \sum \frac{x^n}{n^2} \) on the set \( A = [-1, 1] \) is dominated by the series \( \sum \frac{1}{n^2} \). By the Weierstrass M-test, with \( M_n = \frac{1}{n^2} \), the given series converges uniformly on \( A \) (\( \sum \frac{1}{n^2} \) is a p series with \( p = 2 \)).
CHAPTER V

INFINITE PRODUCTS

Definition 5.1. Given a sequence \( u_n \) of real or complex numbers, let \( p_1 = u_1 \), \( p_2 = u_1 u_2 \), \( p_n = u_1 u_2 \cdots u_n = \prod_{k=1}^{n} u_k \).

A sequence \( \{ p_n \} \) formed in this way is called an infinite product (or simply a product). The number \( q_n \) is called the \( n \)th factor of the \( n \)th partial product \( p_n \).

Note 5.1. The following symbols are used in connection with the above definition:

\[
\prod_{n=1}^{\infty} u_n = \prod_{n=1}^{\infty} u_n.
\]

We also write \( \prod_{n=1}^{\infty} u_n \) when there is no danger of confusion.

The symbol \( \prod_{n=N+1}^{\infty} u_n \) means \( \prod_{n=1}^{\infty} u_{N+n} \).

By analogy with infinite series, it would seem natural to call the product \( \prod_{n=1}^{\infty} u_n \) convergent if \( \{ p_n \} \) converges.

However, this definition would be inconvenient since every product having one factor equal to zero would converge, regardless of the behavior of the remaining factors. The following definition turns out to be more useful:

Definition 5.2. Given an infinite product \( \prod_{n=1}^{\infty} u_n \), let

\[
p_n = \prod_{k=1}^{n} u_k.
\]
(a) If infinitely many factors $u_n$ are zero, we say the product diverges to zero.

(b) If no factor $u_n$ is zero, we say the product converges if there exists a number $p \neq 0$ such that $\{p_n\}$ converges to $p$. In this case, $p$ is called the value of the product and we write $p = \prod_{n=1}^{\infty} u_n$. If $\{p_n\}$ converges to zero, we say the product diverges to zero.

(c) If there exists an $N$ such that $n > N$ implies $u_n \neq 0$ we say $\prod_{n=1}^{\infty} u_n$ converges, provided that $\prod_{n=N+1}^{\infty} u_n$ converges as described in (b). In this case, the value of the product $\prod_{n=1}^{\infty} u_n$ is

$$u_1 u_2 \cdots u_n \prod_{n=N+1}^{\infty} u_n.$$

(d) $\prod_{n=1}^{\infty} u_n$ is called divergent if it does not converge as described in (b) or (c).

**Example 5.1.** $\prod_{n=1}^{\infty} (1 + \frac{1}{n})$ and $\prod_{n=2}^{\infty} (1 - \frac{1}{n})$ are both divergent. In the first, $p_n = n + 1$, and in the second case, $p_n = \frac{1}{n}$.

**Theorem 5.1. (Cauchy's Condition for products).** The infinite product $\prod u_n$ converges, if and only if, for every $\epsilon > 0$ there exists an $N$ such that $n > N$ implies

$$|u_n u_{n+1} \cdots u_{n+k-1}| < \epsilon,$$

for $k = 1, 2, 3, \ldots$.

**Proof:** Assume that the product $\prod u_n$ converges. We assume
that no $u_n$ is zero (discarding a few terms if necessary).

Let $p_n = u_1 \cdots u_n$ and $p = \lim_{n \to \infty} p_n$. Then $p \neq 0$ and hence there exists an $M > 0$ such that $|p_n| < M$. Now $\{p_n\}$ satisfies the Cauchy condition for sequences. Hence, given $\epsilon > 0$, there is an $N$ such that $n > N$ implies $|p_{n+k} - p_n| < \epsilon M$ for $k = 1, 2, 3, \ldots$. Dividing through by $|p_n|$ we obtain

$$|u_n u_{n+1} \cdots u_{n+k} - 1| < \epsilon.$$

Now assume that the given condition (1) holds. Then $n > N$ implies $u_n \neq 0$. Take $\epsilon = \frac{1}{3}$ in (1), let $N_0$ be the corresponding $N$ and let $q_n = u_{N_0} u_{N_0+1} u_{N_0+2} \cdots u_n$, where $n > N_0$. The given condition (1) then implies that $\frac{1}{3} \leq q_n \leq \frac{2}{3}$. Therefore, if $\{q_n\}$ converges, it cannot converge to zero. To show that $\{q_n\}$ does converge, let $\epsilon > 0$ be arbitrary and write the given condition as follows:

$$\frac{|q_{n+k} - q_n|}{q_n} < \epsilon.$$

This gives us $|q_{n+k} - q_n| < \epsilon |q_n| < \frac{2}{3} \epsilon$. Therefore $\{q_n\}$ satisfies the Cauchy condition for sequences and hence is convergent. This means that the product $\prod u_n$ converges.

**Note 5.2.** Taking $k = 1$ in the above given condition, we find that convergence of $\prod u_n$ implies $\lim_{n \to +\infty} u_n = 1$. For this reason, the factors of a product are written as $u_n = 1 + a_n$. Thus the convergence of $\prod (1 + a_n)$ implies $\lim_{n \to +\infty} a_n = 0$.

**Theorem 5.2.** (A necessary and sufficient condition for
convergence of infinite products). Assume that each $a_n > 0$. Then the product $\prod (1 + a_n)$ converges, if and only if, the series $\sum a_n$ converges.

**Proof:** Part of the proof is based on the inequality:

(2) \[ 1 + x \leq e^x. \]

Although (2) holds for all real $x$, we need it only for $x \geq 0$. When $x > 0$ (2) is a simple result of the Mean Value Theorem, which gives us $e^x - 1 = xe^{x_0}$, where $0 < x_0 < x$. Since $e^{x_0} \geq 1$, (2) follows at once from this equation.

Now let $S_n = a_1 + a_2 + \ldots + a_n$, $p_n = (1 + a_1)(1 + a_2)\ldots(1 + a_n)$.

Both sequences $\{S_n\}$ and $\{p_n\}$ are increasing and hence to prove the theorem we need only to show that $\{S_n\}$ is bounded, if and only if, $\{p_n\}$ is bounded.

First the inequality $p_n > S_n$ is obvious. Next, taking $x = a_k$ in (2), $k = 1, 2, 3, \ldots, n$, we find

\[
p_n = (1 + a_1)(1 + a_2)\ldots(1 + a_n) \leq e^{a_1}e^{a_2}\ldots e^{a_n} = e^{a_1 + a_2 + \ldots + a_n} = e^{S_n}.
\]

Hence, $\{S_n\}$ is bounded, if and only if, $\{p_n\}$ is bounded. Note that $\{p_n\}$ cannot converge to zero since $p_n \geq 1$. Note also that $p_n \to \infty$ if $S_n \to \infty$.

**Definition 5.3.** The product $\prod (1 + a_n)$ is said to converge absolutely if $\prod (1 + |a_n|)$ converges.

**Theorem 5.3.** Absolute convergence of $\prod (1 + a_n)$ implies convergence.
Proof: The proof follows easily by use of the Cauchy condition and the inequality
\[ |(1 + a_1)(1 + a_2)\ldots(1 + a_n) - 1| < (1 + |a_1|)(1 + |a_2|)\ldots(1 + |a_n|) - 1. \]

Theorem 5.4. Assume \( a_n \geq 0 \). Then the product \( \Pi (1 - a_n) \) converges, if and only if, the series \( \sum a_n \) converges.

Proof: Convergence of \( \sum a_n \) implies absolute convergence (and hence convergence) of \( \Pi (1 - a_n) \).

To prove the converse, assume that \( \sum a_n \) diverges. If \( \{a_n\} \) does not converge to zero, then \( \Pi (1 - a_n) \) also diverges. Therefore we can assume that \( a_n \to 0 \) as \( n \to \infty \). Discarding a few terms if necessary, we can assume that each \( a_n \leq \frac{1}{2} \). Then each factor \( 1 - a_n \geq \frac{1}{2} \) (and hence \( \neq 0 \)). Let
\[ p_n = (1 - a_1)(1 - a_2)\ldots(1 - a_n), q_n = (1 + a_1)(1 + a_2)\ldots(1 + a_n). \]

Since we have
\[ (1 - a_k)(1 + a_k) = 1 - a_k^2 \leq 1 \]
we can write \( p_n \to q_n \). But in the proof of theorem 5.2, we observed that \( q_n \to \infty \) if \( \sum a_n \) diverges. Therefore, \( p_n \to 0 \) as \( n \to \infty \) and, by part b of definition 5.2, it follows that \( \Pi (1 - a_n) \) diverges to zero.

Example 5.2. The product \( \prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1} \) is absolutely convergent. For observe that
\[ \prod_{n=2}^{\infty} \left( \frac{n^3 - 1}{n^3 + 1} \right) = \prod_{n=2}^{\infty} \left( 1 - \frac{2}{n^3 + 1} \right). \]

But \( \left| \frac{-2}{n^3 + 1} \right| < \left| \frac{2}{n^3} \right| = \frac{2}{n^3} \) which is convergent by the
p-series test for ordinary series. Hence by the comparison test for ordinary series, theorem 5.2, and definition 5.3, the given product is absolutely convergent.
CHAPTER VI

UNIFORM CONVERGENCE OF INFINITE PRODUCTS

**Definition 6.1.** Given a product of the form \( \prod_{n=1}^{\infty} (1 + f_n(x)) \) whose terms are functions of \( x \), we shall define (in complete analogy with the theory of series) an interval of convergence of the product, as an interval \( I \) at every point of which all the functions \( f_n(x) \) are defined and the product itself is convergent.

Thus from the theory of ordinary series (interval of convergence) the products \( \prod_{n=1}^{\infty} (1 - \frac{x^2}{n^2}) \), \( \prod_{n=1}^{\infty} (1 + \frac{x^2}{n^2}) \),
\( \prod_{n=1}^{\infty} (1 + \frac{(-1)^n x}{n^2}) \),\ldots
are convergent for every \( x \).

For every \( x \) in \( I \), the product then has a specific value and therefore defines a determinate function \( F(x) \) in \( I \). We say: the product represents the function \( F(x) \) in \( I \), or, \( F(x) \) is expanded in the given product in \( I \). It will be seen that the fundamental properties of \( f(x) \) also hold in the widest measure for \( F(x) \) as long as the product representing \( F(x) \) is uniformly convergent.

**Definition 6.2.** The product \( \prod (1 + f_n(x)) \) is said to be uniformly convergent in an interval \( J \), if, given \( \epsilon > 0 \), a single number \( N \) depending only on \( \epsilon \) and not on \( x \), can be chosen
so that

\[ \left| (1 + f_{n+1}(x))(1 + f_{n+2}(x)) \ldots (1 + f_{n+k}(x)) - 1 \right| < \epsilon \]

for every \( n > N \), every \( k \geq 1 \) and every \( x \) in \( J \).

**Theorem 6.1.** The product \( \Pi (1 + f_n(x)) \) converges uniformly on any set \( A \), where the series \( \sum |f_n(x)| \) converges uniformly to a bounded sum.

**Proof:** The proof consists of a re-examination of theorem 5.2 and definition 5.3 from the point of view of uniformity. Let \( M \) be the upper bound of the sum \( \sum |f_n(x)| \) in the set being considered. Then

\[ (1 + |f_1(x)|) \ldots (1 + |f_n(x)|) < e^{|f_1(x)| + \ldots} \leq e^M. \]

Let

\[ P_n(x) = \prod_{k=1}^{n} (1 + |f_k(x)|). \]

Then

\[ P_n - P_{n-1} = (1 + |f_1(x)|) \ldots (1 + |f_{n-1}(x)|)|f_n(x)| \]

\[ \leq e^M |f_n(x)|. \]

Hence \( \sum P_n - P_{n-1} \) is uniformly convergent and the result follows from theorem 5.2 and definition 5.3.

**Example 5.3.** \( \Pi (1 + \frac{x^n}{n!}) \) is uniformly convergent on any bounded set \( A \) by Example 3.1 and theorem 6.1.

**Example 5.4.** \( \Pi (1 + e^{-nx} \cos nx) \) converges uniformly on any set that is bounded below by a positive constant by example 4.1 and theorem 6.1.
CHAPTER VII

INFINITE INTEGRALS

We assume here the elementary properties of the Riemann integral of a continuous function and proceed to extend the elementary definition so that under certain circumstances the symbol \( \int_a^b f(x) \, dx \) shall be meaningfull even when the interval of integration is infinite or the function \( f(x) \) is unbounded.

For the sake of conciseness, parentheses will be used in some of the definitions to indicate alternative statements.

**Definition 7.1.** (Improper integrals, finite interval). Let \( f(x) \) be (Riemann) integrable on the interval \([a, b - δ)(a + δ, b]\) for every number \( δ \) such that \( 0 < δ < b - a \), but not integrable on the interval \([a, b]\), and assume that

\[
\lim_{\xi \to 0^+} \int_a^{b-\xi} f(x) \, dx = \left( \lim_{\xi \to 0^+} \int_{a+\xi}^b f(x) \, dx \right)
\]

exists. Under these conditions the improper integral \( \int_a^b f(x) \, dx \) is defined to be this limit

\[
\int_a^b f(x) \, dx = \lim_{\xi \to 0^+} \int_a^{b-\xi} f(x) \, dx \left( \lim_{\xi \to 0^+} \int_{a+\xi}^b f(x) \, dx \right)
\]

If the limit (1) is finite the integral \( \int_a^b f(x) \, dx \) is convergent to this limit and the function \( f(x) \) is said to be improperly
integrable on the half open interval \([a, b)\) \(((a, b])\); if the
limit in (1) is infinite or does not exist, the integral is
divergent.

**Definition 7.2.** Let \([a, b]\) be a given finite interval,
let \(a < c < b\), and let both integrals \(\int_a^c f(x)\,dx\) and \(\int_c^b f(x)\,dx\)
be convergent improper integrals in the sense of definition
7.1. Then the improper integral \(\int_a^b f(x)\,dx\) is convergent and
defined to be:

\[
\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.
\]

If either integral on the right hand side of (2) diverges,
so does \(\int_a^b f(x)\,dx\).

**Example 7.1.** Evaluate \(\int_4^5 \frac{dx}{\sqrt{x - 4}}\).

**Solution:** The integral becomes infinite as \(x \to 4^+\). The
given improper integral has the value

\[
\lim_{\epsilon \to 0^+} \int_4^{4+\epsilon} \frac{dx}{\sqrt{x - 4}} = \lim_{\epsilon \to 0^+} \left[ 2\sqrt{x - 4} \right]_4^{4+\epsilon} = 2
\]

**Example 7.2.** For what values of \(p\) does \(\int_0^1 \frac{dx}{x^p}\) converge?

**Solution:** If \(p \leq 0\), the integral is proper and is convergent.
If \(0 < p < 1\),

\[
\int_0^1 x^{-p} = \lim_{\epsilon \to 0^+} \left[ \frac{x^{1-p}}{1-p} \right]_0^1 = \frac{1}{1-p}, \text{ and the integral}
\]
converges. If \(p = 1\), \(\lim_{\epsilon \to 0^+} \left[ \ln x \right]_\epsilon^1 = \lim_{\epsilon \to 0^+} [-\ln \epsilon]^{-\infty}\), and if
\(p > 1\), \(\lim_{\epsilon \to 0^+} \left[ \frac{x^{1-p}}{1-p} \right]' = +\infty\). Therefore the given integral
converges, if and only if, \(p < 1\).
In order to continue our present development we state without proof the following well known theorem:

Theorem. (Fundamental theorem of integral calculus). Let \( f(x) \) be continuous in the closed interval \([a, b]\) or \([b, a]\) and if \( F(x) \) is any function whose derivative is \( f(x) \) on this interval, so that \( F'(x) = f(x) \); then

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

Theorem 7.1. If \( f(x) \) is continuous in the open interval \((a, b)\), and if there exists a function \( F(x) \) which is continuous over the closed interval \([a, b]\) and such that \( F'(x) = f(x) \) in the open interval, then the integral \( \int_a^b f(x) \, dx \), whether proper or improper, converges and

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

Proof: By the Fundamental Theorem of Integral Calculus, for any \( c \) between \( a \) and \( b \), and sufficiently small positive \( \epsilon \) and \( \eta \),

\[
\int_c^{a+\epsilon} f(x) \, dx + \int_{b-\eta}^b f(x) \, dx = F(b - \eta) - F(a + \epsilon)
\]
and the result follows from the continuity of \( F(x) \) at \( a \) and \( b \).

Example 7.3. In Example 7.1 above, let \( F(x) = 2 \sqrt{x - 4} \). Then

\[
\int_4^5 \frac{dx}{4 \sqrt{x - 4}} = \left[ \frac{1}{2} \sqrt{x - 4} \right]_4^5 = 2.
\]

Definition 7.3. (Improper integral, infinite interval). Let \( f(x) \) be (Riemann) integrable on the interval \([a, u]\) for every \( u > a \), and assume that \( \lim_{u \to \infty} \int_a^u f(x) \, dx \) exists. Under
these conditions the improper integral
\[
\int_{a}^{\infty} f(x) \, dx = \lim_{u \to \infty} \int_{a}^{u} f(x) \, dx.
\]
If the limit in (1) is finite the improper integral is convergent to this limit (3) and the function \( f(x) \) is said to be improperly integrable on the interval \([a, +\infty)\); if the limit (3) is infinite or does not exist, the integral is divergent.

A similar definition holds for the improper integral \( \int_{-\infty}^{a} f(x) \, dx \).

**Definition 7.4.** Let \( f(x) \) be (Riemann) integrable on every finite closed interval, and assume that both improper integrals \( \int_{0}^{\infty} f(x) \, dx \) and \( \int_{-\infty}^{0} f(x) \, dx \) converge. Then the improper integral \( \int_{-\infty}^{\infty} f(x) \, dx \) is convergent and defined to be:

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{0}^{\infty} f(x) \, dx + \int_{-\infty}^{0} f(x) \, dx.
\]
If either integral on the right-hand side of (4) diverges, so does \( \int_{-\infty}^{\infty} f(x) \, dx \).

**Note 7.1.** The improper integral (4) could have been defined unambiguously:
\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{c} f(x) \, dx + \int_{c}^{\infty} f(x) \, dx,
\]
where \( c \) is an arbitrary number.

**Definition 7.5.** Let \( f(x) \) be improperly integrable on the interval \((a, c]\) and on the interval \([c, +\infty)\) where \( c \) is any constant greater than \( a \). Then the improper integral \( \int_{a}^{\infty} f(x) \, dx \) is convergent and defined to be:

\[
\int_{a}^{\infty} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{\infty} f(x) \, dx.
\]
If either integral on the right-hand side of (5) diverges, so does \( \int_{a}^{\infty} f(x) \, dx \).
Similar statements hold for a similarly improper integral $\int_{-\infty}^{a} f(x) \, dx$.

**Example 7.4.** Evaluate $\int_{0}^{+\infty} e^{-ax} \, dx$, $a > 0$.

**Solution:**
$$\int_{0}^{+\infty} e^{-ax} \, dx = \lim_{u \to +\infty} \int_{0}^{u} e^{-ax} \, dx = \lim_{u \to +\infty} \left[ \frac{1}{a} - \frac{e^{-au}}{a} \right] = \frac{1}{a}.$$  

**Example 7.5.** For what values of $p$ does $\int_{0}^{+\infty} \frac{dx}{x^p}$ converge?

**Solution:** For this integral to converge, both $\int_{0}^{1} \frac{dx}{x^p}$ and $\int_{1}^{+\infty} \frac{dx}{x^p}$ must converge. But according to example 7.2, they never converge for the same value of $p$. Hence the answer is none.

**Theorem 7.2.** (Comparison test). If $g(x)$ is a function which dominates a nonnegative function $f(x)$ on the interval $[a, b]$ ($[a, +\infty)$), if both $g$ and $f$ are integrable on $[a, c]$ for every $c$ such that $a < c < b$ ($a < c$), and if the improper integral $\int_{a}^{b} g(x) \, dx$ converges, then so does $\int_{a}^{b} f(x) \, dx$.

**Proof:** Since the two functions are nonnegative, the two integrals $\int_{a}^{c} f(x) \, dx$ and $\int_{a}^{c} g(x) \, dx$ are monotonically increasing functions of $c$. Both have infinite limits as $c \to b$ ($c \to +\infty$). Since $f(x) \leq g(x)$ implies $\lim f(x) \leq \lim g(x)$ then the inequalities
$$0 \leq \int_{a}^{c} f(x) \, dx \leq \int_{a}^{c} g(x) \, dx$$
imply the inequalities
$$0 \leq \int_{a}^{b} f(x) \, dx \leq \int_{a}^{b} g(x) \, dx \quad \left( 0 \leq \int_{a}^{+\infty} f(x) \, dx \leq \int_{a}^{+\infty} g(x) \, dx \right),$$
and the proof is complete.
Example 7.6. Consider the function \( \frac{1}{x^2 + 5x + 17} \).

Observe that for \( x \geq 1 \), this function is less than \( \frac{1}{x^2} \). Hence the convergence of \( \int_1^{+\infty} \frac{dx}{x^2} \) implies that of \( \int_1^{+\infty} \frac{dx}{x^2 + 5x + 17} \).

Note 7.2. If the roles of \( f(x) \) and \( g(x) \) are interchanged in theorem 7.2, we have a comparison test for divergence.

Example 7.7. Since for \( 0 < x \leq 1 \), \( \frac{1}{x^2 + 5x} \leq \frac{1}{6x} \) the divergence of \( \int_0^{1} \frac{dx}{x} \) implies that of \( \int_0^{1} \frac{dx}{x^2 + 5x} \).
CHAPTER VIII

UNIFORM CONVERGENCE OF INFINITE INTEGRALS

**Definition 8.1.** (Uniform convergence). Let \( f(x,y) \) be defined for every point \( x \) of a set \( A \) and every \( y \) of the interval \( I = [c, d] \) \( ([c, +\infty)) \), and assume that for every \( x \) in \( A \) and every \( y \) in \( I \), \( f(x,y) \) as a function of \( y \) is (Riemann) integrable on \([c, B]\). Then the integral

\[
\int_{c}^{d} f(x,y) \, dy = \left( \int_{c}^{B} f(x,y) \, dy \right)
\]

converges uniformly to a function \( F(x) \) for every \( x \) in \( A \), written

\[
\int_{c}^{B} f(x,y) \, dy \Rightarrow F(x), \text{ as } B \to d \ (B \to +\infty),
\]

if and only if, corresponding to every \( \epsilon > 0 \) there exists a number \( \delta \) such that the inequality \( \delta < B < d \ (\delta < B) \) implies

\[
\left| \int_{c}^{B} f(x,y) \, dy - F(x) \right| < \epsilon
\]

for every \( x \) in \( A \). The variable \( x \) is a parameter.

**Note 8.1.** As with infinite series, uniform convergence implies convergence, but not conversely.

**Theorem 8.1.** Under the assumption of definition 8.1 and with the further assumption that (1) converges for each \( x \) in \( A \), the convergence of (1) is uniform, if and only if,

\[
\int_{B}^{d} f(x,y) \, dy \Rightarrow 0 \quad \left( \int_{B}^{+\infty} f(x,y) \, dy \Rightarrow 0 \right),
\]

as \( B \to d \ (B \to +\infty) \); in other words, if and only if, corresponding to \( \epsilon > 0 \) there exists a number \( \delta = \delta(\epsilon) \) belonging to
the interval \((c, d) \ ((c, +\infty))\) and such that the inequality
\(\delta < B < d \ (\delta < B)\) implies
(5) \(\left| \int_B^d F(x,y)\,dy \right| < \varepsilon \left( \left| \int_B^{+\infty} f(x,y)\,dy \right| < \varepsilon \right)\)
for every \(x\) in \(A\).

**Proof**: With the above assumptions, assume also that (1) is uniformly convergent to a function \(F(x)\) on \(A\). Then there exists a number \(\delta = \delta(\varepsilon)\) belonging to the interval \(I\) such that the inequality \(\delta < B < d \ (\delta < B)\) implies
\(\left| \int_0^B f(x,y)\,dy - F(x) \right| < \varepsilon\)
If \(B\) is taken arbitrarily close to \(d\) we have by definition
7.1 (definition 7.3) that
\[\lim_{B \to d} \int_B^d f(x,y)\,dy = \int_c^d f(x,y)\,dy\ (\lim_{B \to \infty} \int_c^B f(x,y)\,dy = \int_c^{+\infty} f(x,y)\,dy).\]
Therefore in the sense of definition 7.1 (definition 7.3)
\(\int_a^d f(x,y)\,dy\) is defined to be \(F(x)\) under our assumptions.

Now recall that
\[\int_c^d f(x,y)\,dy = \int_c^B f(x,y)\,dy + \int_B^d f(x,y)\,dy\]
\(= \left( \int_c^{+\infty} f(x,y)\,dy - \int_c^B f(x,y)\,dy \right)\)
or
\[\int_B^d f(x,y)\,dy = \int_c^d f(x,y)\,dy - \int_c^B f(x,y)\,dy\]
\(= \left( \int_B^{+\infty} f(x,y)\,dy - \int_B^c f(x,y)\,dy \right),\]
so that
\[\left| \int_B^d f(x,y)\,dy \right| = \left| \int_c^B f(x,y)\,dy - F(x) \right| < \varepsilon,\]
Conversely with the given assumptions assume for \( \epsilon > 0 \) there exists a number \( f = f(a) \) belonging to the interval \((c, d)\) ans such that the inequality \( \int B < d \) implies
\[
\left| \int_B^d f(x, y) \, dy \right| < \epsilon \quad \left( \int_B^d f(x, y) \, dy \right),
\]
for every \( x \) in \( A \). By the same argument as above, we have that
\[
\left| \int_c^B f(x, y) \, dy - F(x) \right| = \left| \int_c^B f(x, y) \, dy - \int_c^d f(x, y) \, dy \right|
= \left| \int_B^d f(x, y) \, dy \right| < \epsilon
\]
\[
\left( \left| \int_c^B f(x, y) \, dy - F(x) \right| = \left| \int_c^B f(x, y) \, dy - \int_c^{+\infty} f(x, y) \, dy \right|
= \left| \int_B^d f(x, y) \, dy \right| < \epsilon \right).
\]

**Definition 8.2.** (Negation of uniform convergence). Under the assumption of theorem 8.1 the convergence of (I) fails to be uniform, if and only if, there exists a positive number \( \epsilon \) such that corresponding to an arbitrary number \( f \) belonging to the interval \((c, d)\) \((c, +\infty)\) there exists a number \( B \) of the interval \((d, d)\) \((d, +\infty)\) and a point \( x \) in \( A \) such that
\[
\left| \int_B^d f(x, y) \, dy \right| \geq \epsilon \quad \left( \int_B^{+\infty} f(x, y) \, dy \right) \geq \epsilon
\]

**Example 8.1.** Show that the integral \( \int_0^{+\infty} \frac{\sin xy}{y} \, dy \) converges uniformly for \( x \geq d > 0 \) but is not uniformly convergent for \( x < 0 \).
Definition 8.3. (Dominance). The statement that a function $g(x,y)$ dominates a function $f(x,y)$ for $x$ belonging to $A$ and $y$ belonging to $C$ means that for every $x$ in $A$ and $y$ in $C$

$$|f(x,y)| \leq g(x,y).$$

Theorem 8.2. (Comparison test). Let $f(x,y)$ and $g(x,y)$ be defined for every $x$ of set $A$ and every $y$ of the interval $I = [c, d]$ $([c, +\infty])$ and assume that for every $x$ in $A$ and every $B$ in $I$, $f(x,y)$ and $g(x,y)$ as functions of $y$ are integrable on $[c, B]$. Furthermore, assume that $g(x,y)$ dominates $f(x,y)$ for each $x$ in $A$ and $y$ in $I$, and that

$$\int_c^d g(x,y)dy \geq \int_c^\infty g(x,y)dy$$

converges uniformly if for each $x$ in $A$. Then

$$\int_c^d f(x,y)dy \geq \int_c^\infty f(x,y)dy$$

converges uniformly for each $x$ in $A$.

Proof: By theorem 7.2, (5) is absolutely convergent, and therefore convergent, for each $x$ in $A$. If $\varepsilon > 0$ is given, and if $\delta$ is a point of $I$ such that the inequality $\delta < B < d$ $(\delta < B)$ implies

$$\int_B^d g(x,y)dy < \varepsilon \quad \left(\int_B^\infty g(x,y)dy < \varepsilon\right),$$

for all $x$ in $A$, the appropriate corresponding inequalities for $f(x,y)$ follow from

$$\left|\int_B^{B'} f(x,y)dy\right| \leq \int_B^{B'} |f(x,y)|dy \leq \int_B^d g(x,y)dy,$$

where $B' > B$ (let $B' \to d --$ or $B' \to +\infty$). Q. E. D.

Since an integral of the form $\int_c^d M(y)dy \left(\int_c^\infty M(y)dy\right)$,
where the integral is independent of $x$, converges uniformly in any set $A$, whenever it converges at all. We have as a special case of theorem 8.2 the analogue of the Weierstrass M-test for infinite series.

**Theorem 8.3.** (Weierstrass M-test). Let $f(x,y)$ and $M(y)$ be defined for every $x$ in $A$ any $y$ in $I = [c, d]$ ($[c, +\infty)$), and assume that for every $x$ in $A$ and every $B$ in $I$, $f(x,y)$ and $M(y)$ as functions of $y$ are integrable on $[c, d]$. Furthermore, assume that for every $x$ in $A$

$$(7) \quad |f(x,y)| \leq M(y)$$

and that

$$(8) \quad \int_c^d M(y)dy \left( \int_c^{+\infty} M(y)dy \right)$$

is convergent. Then

$$\int_c^d f(x,y)dy \left( \int_c^{+\infty} f(x,y)dy \right)$$

converges uniformly on $A$.

**Proof:** The proof is analogous with the proof of theorem 8.2.

**Example 8.2.** Show that $\int_0^{+\infty} e^{-y}\cos xydy$ converges uniformly for all real $x$.

**Solution:** $|e^{-y}\cos xy| \leq e^{-y}$ and $\int_0^{+\infty} e^{-y}dy$ converges. Taking $M(y) = e^{-y}$ the solution follows from theorem 8.3.
BIBLIOGRAPHY


