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A comprehensive study of Hilbert spaces

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ABSTRACT

MATHEMATICS

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A COMPREHENSIVE STUDY OF HILBERT SPACES

Advisor: Dr. Lloyd K. Williams

Thesis dated August 1971

A Hilbert Space $H$ is an inner product space which as a metric space is complete. The most important theorems proved are Riesz's Representation Theorem, theorems of Adjoints and operators. Measure of Spectrum and spectral integrals are briefly discussed.
A COMPREHENSIVE STUDY OF HILBERT SPACES

A THESIS
SUBMITTED TO THE FACULTY OF ATLANTA UNIVERSITY
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BY
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DEPARTMENT OF MATHEMATICS

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K. J. S.
INTRODUCTION

This report presents a comprehensive study of the Hilbert-Spaces. It is divided into three parts. The first part deals with the preliminaries and relations required to understand Hilbert Spaces itself. A few important theorems have been proved, and the last part introduces the spectrum of an operator, spectral measures and integrals.

The author is indebted to Professor Lloyd K. Williams for the guidance during the project and preparation of this thesis. The author also wishes to acknowledge his thanks to Professor Nazir A. Warsi.
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LINEAR FUNCTIONALS:
Let $V$ be a vector space over a field $K$. A mapping $\phi: V \to K$ is termed a linear functional if for every $u, v \in V$ and every $a, b \in K$, $\phi(au + bv) = a\phi(u) + b\phi(v)$. In other words, a linear functional on $V$ is a linear mapping from $V$ into $K$.

BI-LINEAR FUNCTIONAL:
A bi-linear functional is a mapping of $\phi: V \times V \to K$ which satisfies:

(i) $\phi(au_1 + bu_2, v) = a\phi(u_1, v) + b\phi(u_2, v)$
(ii) $\phi(u, av_1 + bv_2) = a\phi(u, v_1) + b\phi(u, v_2)$

For all $a, b \in K$ and all $u, v \in V$ (i) is expressed by saying $\phi$ is linear in the first variable and (ii) by saying that $\phi$ is linear in the second variable.

QUADRATIC FORMS:
The quadratic form $\bar{\phi}$ induced by a bi-linear functional on a complex vector space is the function defined for each vector $x$ by $\bar{\phi}(x) = \phi(x, x)$.

INNER PRODUCT:
A complex vector space $H$ is called an inner product space if to each ordered pair of vectors $x, y \in H$ there is associated
a complex number \((x, y)\), the so called "inner product" of \(x\) and \(y\) such that follows:

(i) \((x, y) = (y, x)\) \(\bar{\text{bar}}\) denotes complex conjugation

(ii) \((x, y, z) = (x, z) + (y, z)\) if \(x, y, z \in \mathbb{H}\)

(iii) \((\alpha x, y) = \alpha (x, y)\) if \(x, y \in \mathbb{H}\), \(\alpha\) is scalar

(iv) \((x, x) \geq 0\) for all \(x \in \mathbb{H}\)

(v) \((x, x) = 0\) iff \(x = 0\)

**NORM OF A VECTOR:**

**Definition 1.** The norm (or length) of a vector \(\vec{x}\), defined \(\|\vec{x}\|\), is the non negative real number defined as:

\[
\|\vec{x}\| = \sqrt{(x, x)}
\]

**Theorem 1.** A necessary and sufficient condition that \(x = 0\) is that \((x, y) = 0\) for all \(y\).

**Proof.** If \((x, y) = 0\) for all \(y\), then in particular \((x, x) = 0\) and consequently since the inner product is strictly positive, \(x = 0\) consequently \(x = 0\), then \((x, y) = (0x, y) = 0(x, y) = 0\).

**Theorem 2.** For any vectors \(x\) and \(y\), \(\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2\).

**Proof.** \(\|x + y\|^2 = (x+y, x+y) = (x,x) + (x,y) + (y,x) + (y,y) = \|x\|^2 + \|x\|\|y\| + \|y\|\|x\| + \|y\|^2\)

Similarly \(\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| + \|y\|^2\).

Adding we get \(\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2\).

**Definition 2.** \(x\) is orthogonal to \(y\) \((x \perp y)\) if \((x, y) = 0\). For other orthogonal vectors, \(\|x + y\|^2 = \|x\|^2 + \|y\|^2\).
A family $\{x_j\}$ of vectors is an orthogonal family if $x_j \perp x_k$ when $j \neq k$.

**INEQUALITIES OF BESSEL AND SCHWARZ:**

A vector $x$ is normalized, or is a unit vector if $\|x\| = 1$, the process of replacing a non-zero vector $x$ by the unit vector $x/\|x\|$ is called normalization.

A family $\{x_j\}$ of vectors is an orthonormal family if it is orthogonal family and each vector is normalized or more explicitly if $(x_j, x_k) = \delta_{jk} \forall j, k$.

**Theorem 3. (Bessel's inequality)** If $\{x_j\}$ is a finite orthonormal family of vectors then $\sum_j (x, x_j)^2 \leq \|x\|^2$ for every vector $x$.

**Proof.**

\[
0 \leq 11x - \sum_{j=1}^{\infty} (x, x_j)x_j^2 \\
= (x, x) - (x, \sum_{j=1}^{\infty} (x, x_j)x_j) - (\sum_{j=1}^{\infty} (x, x_j)x_j, x) \\
+ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (x, x_j)x_j, (x, x_j)x_j \\
= 11x^2 - \sum_{j=1}^{\infty} ((x, x_j), (x, x_j)) - \sum_{j=1}^{\infty} (x, x_j)(x, x_j) \\
= 11x^2 - 2 \sum_{j=1}^{\infty} 1(x, x_j) + \sum_{j=1}^{\infty} |(x, x_j)|^2 \\
= 11x^2 - \sum_{j=1}^{\infty} 1x_j^2 \\
\sum_{j=1}^{\infty} 1(x, x_j)^2 \leq 11x^2
\]

**Theorem 4. (Cauchy-Schwarz inequality)** In a Hilbert space $H$, $|1(x, y)| \leq \|x\| \|y\| \forall x, y$

**Proof.** If $x = 0$ or $y = 0$, result is obvious. If $x \neq 0$, $y \neq 0$ for any
complex number \( A \)

\[
0 \leq (x+\lambda y, x+\lambda y)
\]

\[
= (x, x) + (x, \lambda y) + (\lambda y, x) + (\lambda y, \lambda y)
\]

\[
= 111^2 + y^2 (x, y) + \lambda (y, x) + \lambda^2 (y, y)
\]

let \( l = (x, y) \)

\[
= 111^2 - \frac{(x, y)(x, y)(y, y)(y, x)}{(y, y)(y, y)} + \frac{|(x, y)|^2 (y, y)}{(y, y)^2} + \frac{|(x, y)|^2 (y, y)}{(y, y)^2}
\]

\[
= 111^2 - \frac{2|(x, y)|^2 + |(x, y)|^2}{(y, y)(y, y)}
\]

\[
= 111^2 - \frac{|(x, y)|^2}{(y, y)}
\]

\[
(\therefore) \quad \frac{|(x, y)|^2}{(y, y)} \leq 111^2
\]

or \( |(x, y)| \leq 111 11y \)

or \( |(x, y)| \leq 111 11y \)
CHAPTER II

HILBERT SPACES

**Theorem 5.** The norm in an inner product is:

(i) strictly positive ($\|x\| > 0$ when $x \neq 0$)

(ii) positively homogeneous ($\|\lambda x\| = |\lambda| \|x\|$)

(iii) subadditive ($\|x+y\| \leq \|x\| + \|y\|$)

**Proof.** (i) The strict positiveness of the norm is a restatement of the strict positiveness of the inner product.

(ii) $\|\alpha x\|^2 = (\alpha x, \alpha x) = \alpha^2 (x, x) = |\alpha|^2 \|x\|^2$

or $\|\alpha x\| = |\alpha| \|x\|

(iii) $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + (x, y) + (x, y)$

$= \|x\|^2 + \|y\|^2 + 2\rho(x, y)$ where $\rho$ is the real part.

$\leq \|x\|^2 + \|y\|^2 + 2 \|x\| \|y\| \text{ Cauchy-Schwarz inequality.}$

$\leq \|x\|^2 + \|y\|^2 + 2 \|x\| \|y\|$

$= (\|x\| + \|y\|)^2$

$\therefore \|x+y\| \leq \|x\| + \|y\|$

**Theorem 6.** If the distance from a vector $x$ to a vector $y$ is defined to be $\|x-y\|$, then with respect to this distance function, $H$ is a metric space.

**Proof.** The fact that distance function is strictly positive
(ω11x-y11≥0) follows from strict positiveness of the norm.
The fact that the distance function is symmetric (ω11x-y11=11y-x11
for every pair of vectors x and y) follows from the positive homogenety of the norm and identity (x-y)=(-1)(y-x). The validity of the triangle inequality (11x-y11≤11x-z11+11z-y11 for every infinite vectors x,y,z) follows from subadditivety of the norm and the identity x-y=(x-z)+(z-y).

In view of this theorem for inner product spaces, all such topological concepts as consequence continuity, seperability close set, closus of a set, closed set and all such metric concepts as uniform continuity, Cauchy sequence and completeness can be used.

A Hilbert Space H is an inner product space which as a metric space is complete.

A normed vector space is a vector space with a strictly positive, positively homogeneous, and subadditive norm. A Banach Space is a normed vector space which, as a metric space is complete.

SUBSPACES:

A linear manifold is a non-empty subset M of H such that if x and y are in M then Lx+Bx∈M for every pair of complex numbers L and B. A subspace is a closed linear manifold.

Example 1. Consider a set D containing 0 only then D=∅0
Let \( L, B \) be any complex number, then \( L_0B_0 = 0 + 0 = 0 \)
\[
\therefore L_0B_0 \in \mathbb{D} \text{ is a linear manifold.}
\]
To show \( \mathbb{D} \) is closed, let \( x_n \in \mathbb{D} \rightarrow x_n \xrightarrow{\text{lim}} x_0 \)
\[
(\text{lim } x_n, y) = \text{lim}(x_n, y) = 0
\]
\[
\therefore x_0 \in \mathbb{D} = 0.
\]
**Definition 2.** The subspace spanned by an arbitrary subset \( M \) of \( H \) (the span of \( M \), in symbols \( \text{VM} \)) as the intersection of all subspaces containing \( M \), or equivalently, as the least subspace containing \( M \).

**VECTORS IN AND OUT OF SUBSPACES:**

A vector \( \vec{x} \) is orthogonal to a subset \( M \) of \( H \), in symbols \( \vec{x} \perp M \), if \( \vec{x} \perp y \) for all \( y \in M \). Two important results can be obtained from the definition. The first result is that the minimum of the distances from any fixed vector to the vectors of any fixed subspace is always obtained. The second result is essentially that if a subspace is a proper subset of \( H \), then there exists a non zero vector orthogonal to the subspace.

**Theorem 7.** Let \( M \) be a closed subspace in \( H \), and if \( d = \text{inf} \left\{ \| y - x_0 \| : y \in M \right\} \) then there exists a vector \( y_0 \) in \( M \) such that \( \| y_0 - x_0 \| = d \).

**Proof.** Let \( y \) be the sequence of vectors in \( M \) such that \( \| y_n - x_0 \| \rightarrow d \) as \( n \rightarrow \infty \)
the law of parallelogram states:
\[ 11x+y_11^2 + 11x-y_11^2 = 111x_11 + 211y_11^2 \]

Putting \( x = x_0 - y_n \)

\[ y = x_0 - y_m \]

we get

\[ 411x_0 \frac{1}{2} (y_n + y_m) + 11y_n - y_m 11^2 = 211x_0 - y_n 11^2 + 211x_0 - y_n 11^2 \rightarrow yd^2 \]

as \( n, m \rightarrow \infty \)

since \( (y_n + y_m) \subseteq M \) it follows that

\[ 111 \frac{1}{2} (y_n + y_m) - x11 \nRightarrow yd^2 \]

hence \( 11y_n - y_m 11 \rightarrow 0 \) if \( m, n \rightarrow \infty \)

so that \( \{y_n\} \) is a Cauchy sequence

since \( H \) is complete, \( M \) is closed

\[ \lim y_n = y_0 \text{ exists} \subseteq M \]

hence \( 11x_0 - y_0 11 = \lim 11x_0 - y_n 11 = d. \)

**Theorem 8.** If \( M \) and \( N \) are subspaces such that \( M \subseteq N \) and \( M \neq N \), then there exists a non zero vector \( z \) in \( N \) such that \( z \perp M. \)

**Proof.** Let \( x \) be any vector in \( N \) which is not in \( M \). Writing

\[ d = \inf \{11y - x11; y \in M\}. \]

By theorem 1 there exists a vector \( y_0 \) in \( M \) such that \( 11y_0 - x11 = d \), writing \( z = y_0 - x, x \notin M \) and due to this fact \( z \neq 0 \). Since \( y_0 + Ly \in M \) for every vector \( y \) in \( M \), for every complex number \( L \), it follows:

\[ 11z + Ly11 = 11(y_0 + Ly) - x11 \geq d \] and hence

\[ 0 \leq 11z + Ly11^2 - 11z11^2 = L(z, y) + L(y, z) + L11y11^2 \]

If in particular \( L = B(z, y) \) for any real \( B \), then
\[ 0 \leq B(z, y)(z, y) + B(z, y)(y, z) + B^2(z, y)^2 11y1^2 \]

\[ 0 \leq 2B \left| (y, z) \right|^2 + B^2 \left| (y, z) \right|^2 11y1^2. \]

The validity of this inequality for small negative values of B implies the vanishing of the co-efficient of the linear term.

It is concluded that \( z \perp y \) hence, since y is an arbitrary vector in M that \( z \perp M \).

**ORTHOGONAL COMPLEMENTS:**

The orthogonal complement of a subset M of H in symbols \( M^\perp \) is the set of all the vectors x such that \( x \perp M \). If M and N are subspaces such that \( M \subset N \) the orthogonal complement of M in N is the set \( N \cap M^\perp \).

**Theorem 9.** If M is a subset of H, then M is a subspace and \( M \cap M^\perp = \{0\} \).

**Proof.** If \( x \notin M \) and \( y_1 \), and \( y_2 \) are in \( M^\perp \), then for every pair of complex numbers \( L_1 \) and \( L_2 \)

\[ (x, L_1 y_1 + L_2 y_2) = L_1 \overline{(x, y_1)} + L_2 \overline{(x, y_2)} = 0 \]

so that M is a linear manifold and the fact M is closed.

Let \( y_\infty \in M \) \( \forall \) \( y_\infty \rightarrow y_\circ \)

then

\[ (y_\circ , x) = (\lim y_\infty , x) = \lim (y_\infty , x) = 0 \]

\[ \therefore y_\circ \text{ is an element of } M^\perp \]

To show \( M \cap M^\perp = \{0\} \).

Let \( x \in M \) and \( x \in M^\perp \).
\[(x,x)=0 \Rightarrow x=0\]

\[\therefore M \cap M^\perp = \{0\}\]

**Theorem 9.** If \(M\) and \(N\) are orthogonal subspaces, then \(M+N\) is closed.

**Proof.** Suppose that \(\{z_n\}\) is a sequence of vectors of \(M+N\) so that for each \(n\), \(z_n = x_n + y_n\) with \(x_n \in M, y_n \in N\) and suppose that the sequence \(\{z_n\}\) converges to a vector \(z\) in \(H\). By Pythagorean theorem:

\[11z_n - z_m, 11^2 = 11x_n - x_m, 11^2 + 11y_n - y_m, 11^2\]

for every \(n\) and \(m\), and therefore both sequences \(x_n\) and \(y_n\) are Cauchy. If \(x_n \rightarrow x\) and \(y_n \rightarrow y\) then \(x \in M\) and \(y \in N\).

**FROM THE CONTINUITY OF ADDITION:**

\[z_n \rightarrow x + y \quad \text{hence} \quad z \in M + N.\]

**Theorem 10.** If \(M\) is a subspace, then \(M + M^\perp = H\).

**Proof.** If \(M + M^\perp = N\) then by theorem 9, \(N\) is a subspace. Since \(M \subseteq N\) and \(M^\perp \subseteq N\) it follows \(N^\perp \subseteq M^\perp\) and therefore \(N^\perp = 0\), \(\therefore N = N^\perp = 0 = H\).

**Definition 4.** The dimension of a subspace \(M\) is defined as the common power of all bases of \(M\). An isomorphism from a Hilbert space \(H\) and a Hilbert space \(R\) is a one-to-one linear transformation \(U\) from \(H\) onto \(R\) such that \((Ux, Uy) = (x, y)\) for every pair of vectors \(x\) and \(y\) in \(H\), an isomorphism from a Hilbert space \(H\) to a Hilbert space \(R\) is a linear transformation \(U\) from \(H\) into \(R\) such that \(11Ux, 11 = 11x, 11\) for every vector \(x\) in \(H\).
Isometry presents not only norms, but all distances $\|U \cdot -U\| = \|U(x-y)\| = \|x-y\|$.

**Theorem 11.** A linear transformation $U$ from a Hilbert space $H$ to a Hilbert space $R$ is an isomorphism iff it is an isometry mapping $H$ and $R$.

**Proof.** We have already seen that an isomorphism is an isometry. If, consequently, $U$ is an isometry and $Ux=Uy$ then $0 = \|U(x-y)\| = \|x-y\|$ if $U$ is one-to-one. The fact that $U$ preserves inner products follows from the assumption that if $\phi(x,y) = (Ux, Uy)$ and $\psi(x,y) = (x,y)$, then the bi-linear functional $\phi$ and $\psi$ induce the same quadratic form. Two Hilbert spaces are called isomorphic if there exists an isomorphism between them.

**BOUNDEDNESS:**

**Definition 5.** A linear transformation $A$ from a Hilbert space $H$ to a Hilbert space $R$ is bounded if there exists a positive real number such that $\|Ax\| \leq \alpha \|x\|$ for all $x \in H$ in symbols $\|A\|$ is the transformation of all such values of $\alpha$.

**Theorem 12.** A linear transformation $A$ from a Hilbert space $H$ to a Hilbert space $R$ is bounded iff it is continuous.

**Proof.** $A: H \rightarrow R$

let $A$ be bounded

then $\|Ax\| \leq \alpha \|x\| \forall x$

let $x_n \rightarrow 0$
\[11Ax_n,11 \leq \alpha 11x_n,11 \rightarrow 0\]
\[\Rightarrow \quad Ax_n \rightarrow 0\]

Now let \(A\) be continuous. If \(A\) is not bounded no \(\alpha\) satisfying the inequality.

Taking \(\alpha = n \exists x_n \Rightarrow 11Ax_n,11 \geq n11x_n,11\)

let \(x_n = Y_n\)

\[n11Yn11\]

\[11Ax_n,11 = \frac{1}{n11Yn11}\]

\[\frac{11AYn11}{\frac{1}{n11Yn11}} \geq \frac{1}{n11Yn11} \geq 1\]

which contradicts hypotheses that \(A\) is continuous.

**Theorem 13.** (Riesz representation theorem). For every bounded linear functional \(x^*\) on a Hilbert space \(H\) there exists a unique element \(Y\) of \(H\) such that \(x^*(x) = (x, y)\) for all \(x \in H\).

**Proof.** Let \(x^*\) be bounded linear functional \(N\) be its null space ie \(N = \{x : x^*(x) = 0\}\) then \(N\) is a closed linear subspace.

To show subspace closed,

let \(\{x_n\}\) be in \(N \Rightarrow x_n \rightarrow x\) then \(x^*(x_n) = 0\)

and \(x^*(x) = x^*(1 \lim x_n) = 1 \lim x^*(x) = 0\)

this makes \(N\) closed as \(0 \in N\)

now if \(N = H\), then \(x^* = 0 \Downarrow x^*(x) = (x, 0)\)

if \(N \neq H\), then by theorem 8 \(N^\perp\) contains non zero vectors \(\alpha\)

let \(\alpha = x^*(z) \neq 0\)
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Let $x \in H$ and look at $x - x^\ast(x)z$

$$x^\ast(x - x^\ast(x)z) = x^\ast(x) - x^\ast(x)x^\ast(z) = 0$$

\[\therefore x - x^\ast(x)z \in N\]

$x - x^\ast(x)z = 0$

$\therefore (x, z) - x^\ast(x)(z, z) = 0$

Let $Y = \frac{-x^\ast(z)}{(z, z)}$ $(z, z)$

Then

$$(x, y) = \left( x, \frac{-x^\ast z}{(z, z)} \right) = \frac{x^\ast(x)}{(z, z)}$$

$$= \frac{x^\ast(x)(z, z)}{(z, z)} = x^\ast(x)$$

$(z, z)$

To show that $Y$ is unique

Let $\exists Y_1 \ni x^\ast(x) = (x, y_1) \forall x$

\[\therefore (x, y_1) - (x, y) = 0 \quad \forall x \]

$$(x, y_1 - y) = 0 \quad \forall x$$

since it is true

Let $x = Y_1 - Y$

$$(Y_1 - Y, Y_1 - Y) = 0$$

$Y_1 - Y = 0$ or $Y_1 = Y$

**Definition 6.** A bi-linear functional $\Phi$ is bounded if there exists a positive real number $\alpha$ such that $|\Phi(x, y)| \leq \alpha \|x\| \|y\|$ for every pair of vectors $x$ and $y$ in $H$ and defining the norm of $\Phi$
in symbols $11\phi 11$ as the transformation of all such values of $\alpha$.

**Definition 7.** A quadratic form $\phi^N$ is bounded if there exists a positive real number $\alpha$ such that $|\phi^N(x)| \leq \alpha 11x11^2$ for all $x$ in $H$.

**Definition 8.** An operator is a bounded linear transformation from $H$ into $H$.

**Theorem 14.** If $A$ and $B$ are operators and if for every vector $x$ and for every complex number $\alpha$,

$$\alpha(A)x = \alpha(Ax), \quad (A+B)x = Ax + Bx$$

and $(AB)x = A(Bx)$, then $\alpha A$, $A+B$ and $AB$ are operators such that

$$11\alpha A11 = |\alpha| \cdot 11A11$$

$$11A+B11 \leq 11A11 + 11B11$$

and $11AB11 \leq 11A11 \cdot 11B11$.

**Proof.** Proof is obvious.

**Theorem 15.** (continuity of an operator) If $A$ is an operator, $x$ is a vector, and $\{x_J\}$ is a family of vectors such that $\sum_J x_J = x$ then $\sum_J Ax_J = Ax$.

**Proof.** For any positive number $\epsilon$ we may find a finite set $J_0$ of indices such that $11x - \sum_J x_J 11 \leq \epsilon$ whenever $J$ is a finite set of indices containing $J_0$. It follows that $11Ax - \sum_J Ax_J 11 \leq 11A11 \epsilon$ whenever $J \supset J_0$. Which implies that $\sum_J Ax_J = Ax$.

**Definition 9.** An operator $A$ is invertible if there exists an operator $B$ such that $AB = BA = 1$.

**Definition 10.** The range of an operator $A$ is the set of all vectors of the form $Ax$, the range of an operator is always a
linear manifold, but it is not necessarily a subspace.

**Theorem 16.** An operator $A$ is invertible if and only if its range is dense in $H$ and there exists a positive real number such that $\|Ax\| \geq \alpha \|x\|$ for every vector $x$.

**Proof.** If $A$ is invertible and if $y \in H$ write $x = A^{-1}y$ since $Ax = y$ it follows that the range of $A$ is not only dense in $H$, but in fact coincides with $H$. It follows also that for every vector $x$,

$$\|x\| = \|A^{-1}Ax\| < \|A^{-1}\| \|x\|$$

that condition of the theorem is satisfied with $\alpha = \frac{1}{\|A^{-1}\|}$.

Suppose now that the range of $A$ is dense and that $\|Ax\| \geq \alpha \|x\|$. By theorem 16 we conclude that the range of $A$ is in fact equal to $H$. If $Ax_1 = Ax_2$ then $0 = \|Ax_1 - Ax_2\| \geq \|x_1 - x_2\|$ and therefore $x_1 = x_2$. This implies that not only is it true that every vector $y$ in $H$ has the form $Ax$ for some $x$ in $H$, but in fact there is exactly one such $x$ and a single valued transformation $B$ of $H$ which itself is defined by writing $By = x$. It is easily defined to be linear and since

$$\|y\| = \|Ax\| \geq \alpha \|x\| = \alpha \|By\|$$

it follows that $B$ is the operator.

**ADJONITS:**

If $A$ is a linear transformation from $H$ into $H$ and if $\Phi(x, y) = \langle Ax, y \rangle$ for every pair of vectors $x$ and $y$, then $\Phi$ is a linear functional. The elementary properties of the inner
product implies that if $A_1 + A_2$ are two linear transformations from $H$ into $H$ such that $(A_1 x, y) = (A_2 x, y)$ for all $x$ and $y$, then $A_1 = A_2$. We now see the connection between linear transformations and bi-linear functionals.

**Theorem 17.** If $A$ is an operator and if $\phi(x, y) = (Ax, y)$ for all $x$ and $y$, then $\phi$ is a bounded bi-linear functional and $\|\phi\| \leq \|A\|$. If consequently $\phi$ is a bounded bi-linear functional, then there exists a unique operator $A$ such that $\phi(x, y) = (Ax, y)$ for all $x$ and $y$.

**Proof.** If $A$ is an operator that if $\phi(x, y) = (Ax, y)$ then $|\phi(x, y)| \leq \|A\| \|x\| \|y\|$ for all $x$ and $y$ and consequently $\|\phi\| \leq \|A\|$. If consequently, $\phi$ is a bounded bi-linear functional and if $\eta_x(y) = \phi(x, y)$ for all $x$ and $y$, then for each fixed $x$, $\eta_x$ is a bounded linear functional. It follows from Riesz' representation theorem that there exists a unique vector $Ax$ such that $\phi(x, y) = (Ax, y)$ for all $y$. The linearity of the transformation $A$ thereby defined is easily defined, its uniqueness follows:

$11Ax11 = (Ax, Ax) = \phi(x, Ax) \leq \|\phi\| \|x\| \|Ax\|$ it follows that $11Ax11 \leq \|\phi\| \|x\| \|Ax\|$ for all $x$. But this implies that $A$ is bounded and $\|A\| \leq \|\phi\|$.  

**Theorem 18.** If $A$ is an operator, then there exists a unique operator $A^*$ called the Adjoinits of $A$, such that $(Ax, y) = (x, A^* y)$.
for all $x$ and $y$, $A^*$ is such that $11A^*11 = 11A11$.

**Proof.** Write $\varphi(x, y) = \langle Ax, y \rangle$ and $\psi(x, y) = \varphi^*(y, x)$ for all $x$ and $y$. By theorem 17 is a bounded bi-linear functional and since this implies that $\psi$ is a bounded bi-linear functional with $11\psi11 = 11\varphi11 = 11A11$, it follows from the conversement of theorem 17 that there exists an operator $A^*$ such that $\psi(x, y) = \langle A^*x, y \rangle$ for all $x$ and $y$ and that $A^*$ is such that $11A^*11 = 11\psi11 = 11A11$. Since $A^*$ is unique the proof is complete by

$$\langle Ax, y \rangle = \varphi(x, y) = \psi^*(y, x) = \langle A^*y, x \rangle = \langle x, A^*y \rangle.$$ 

**Theorem 19.** If $A$ and $B$ are operators and $\alpha$ is a complex number then:

(i) $A^{**} = A$

(ii) $(\alpha A)^* = \alpha A^*$

(iii) $(A + B)^* = A^* + B^*$

(iv) $(AB)^* = B^* A^*$

(v) If $A$ is invertible, then $A^*$ is invertible and $(A^*)^{-1} = (A^{-1})^*$.  

**Proof.** Proof is obvious.

**Definition 11.** A subspace $M$ is invariant under an operator $A$ if $AM \subseteq M$ i.e. if $Ax \in M$ whenever $x \in M$, a subspace $M$ reduces an operator $A$ if both $M$ and $M^\perp$ are invariant under $A$.

**Theorem 20.** A necessary and sufficient condition that a subspace $M$ be invariant under an operator $A$ is that $M^\perp$ be invariant under $A^*$. 
Proof. By symmetry, it is sufficient to prove that the condition is necessary. If $M$ is invariant under $A$ and if $x \in M$ and $y \in M^\perp$, then $(x, A^* y) = (Ax, y) = 0$ so that $A^* y \in M^\perp$ and consequently $M^\perp$ is invariant under $A^*$.

**Definition 12.** An operator $A$ is Hermitian if $A = A^*$. 

**Theorem 21.** A necessary and sufficient condition that an operator $A$ be Hermitian is that the bi-linear functional $\Phi$, defined every pair of vectors $x$ and $y$ by $\Phi(x, y) = A(x, y)$ be symmetric.

**Proof.** A necessary and sufficient condition that $\Phi(x, y) = \Phi(y, x)$ for all $x$ and $y$ is that $(Ax, y) = (y, A^* x)^* = (A^* x, y)$ for all $x$ and $y$. An operator $A$ is Hermitian if $(Ax, x)$ is real for every vector $x$, if $A$ is Hermitian, then $\|A\| = \sup \left\{ |(Ax, x)| : \|x\| = 1 \right\}$. Most of the algebraic properties of the set of Hermitian operators follows from definitions. For instance a real scalar multiple of an Hermitian operator and the sum of two Hermitian operators are Hermitian and that the issue of an invertible Hermitian operator is also Hermitian.

**Definition 13.** An operator $A$ commutes with an operator $B$ ($A \leftrightarrow B$) if $AB = BA$.

**Theorem 22.** The product of two Hermitian operators $A$ and $B$ is Hermitian if $A \leftrightarrow B$. 

Proof. Since \((AB)^* = BA\), the equations \((AB)^* = AB\) and \(BA = AB\) are obviously equivalent. It is concluded from the above theorem that if \(A\) is a Hermitian operator and \(P\) is a real polynomial, then \(P(A)\) is Hermitian.

**NUMERICAL AND UNITARY OPERATORS:**

An operator is called numerical if \(A \leftrightarrow A^*\) if \(A = B + ic\), with \(B\) and \(C\) Hermitian.

**Theorem 23.** A necessary and sufficient condition that operator \(A\) be normal is that:

\[ ||Ax|| = ||A^*x|| \text{ for every vector } x.\]

**Proof.** Since \( ||Ax|| = (Ax, Ax) = (A^*Ax, x) \)

similarly \( ||A^*x|| = (A^*x, A^*x) = (AA^*x, x). \)

The operators \(U\) which satisfy the equations \(UU^* = U^*U = 1\) such operators are called unitary.

**Definition 14.** The projection on a subspace \(M\) is the transformation \(P\) defined for every vector \(x\) of the form \(x + y\), with \(x \in M\) and \(y \in M^\perp\) by \(Pz = x\).

**Theorem 24.** The projection \(P\) on a subspace \(M\) is an indempotent \((P^2 = P)\) and Hermitian \((P^* = P)\) operator, if \(M \neq 0\), then \(||P|| = 1\).

**Proof.** Proof is obvious.

**Theorem 25.** If \(P\) is the projection on a subspace \(M\) and if \(M_1 = \{x: Px = x\}\) and \(M_2\) is the range of \(P\) then \(M_1 = M_2 = M\).

**Proof.** It follows immediately from the definitions of \(M_1, M_2\)
and $P$ that $M_1 \subseteq M_2 \subseteq M$. If on the other hand $x \in M$, then $Px = x$ so that $M \subseteq M_1$ and consequently all these inclusion relations reduce equations.

**Theorem 26.** If $P$ is the projection on a subspace $M$ and if $x$ is a vector such that $\|Px\| = \|x\|$, then $Px = P$.

**Proof.** Since $x = Px + (x - Px)$ and $x - Px \perp M$, it follows

$$\|x\|^2 = \|Px\|^2 + \|x - Px\|^2.$$  

The fact that $\|Px\| = \|x\|$ implies

$$\|x - Px\| = 0.$$  

**Theorem 27.** If $P$ is a projection, then $(Px, x) = \|Px\|^2$ for every vector $x$.

**Proof.** $(Px, x) = (P^2x, x) = (Px, P^*x) = (Px, Px) = \|Px\|^2$. 
CHAPTER III

SPECTRUM OF AN OPERATOR

Definition 14. The spectrum of an operator $A$ in symbols $\sigma(A)$ is the set of all those complex numbers $\lambda$ for which $A - \lambda I$ is not invertible. If $H$ is finite-dimensional, then a necessary and sufficient condition that our operator to be invertible is the vanishing of its determinant a concept which makes no sense in general, not necessarily finite dimensional case. Since the determinant of $A - \lambda I$ is a polynomial in $\lambda$ whose zeroes are exactly the proper values of $A$, it follows that in the finite dimensional case the spectrum of an operator is exactly the set of its value.

Definition 15. A complex number $\lambda$ is a proper value of an operator $A$ if there exists a non zero vector $x$ such that $Ax = \lambda x$. Consequently $\lambda$ is a proper value of $A$ if there exists a unit vector $x$ such that $\|Ax - \lambda x\| = 0$.

Definition 16. A complex number $\lambda$ is an approximate proper value of an operator $A$ if for every finite number $\varepsilon$ there exists a unit vector $x$ such that $\|Ax - \lambda x\| < \varepsilon$.

Definition 17. The approximate point spectrum of an operator $A$ in symbols $\mathcal{P}(A)$ is the set of approximate proper values of $A$.

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Theorem 28. If $A$ is an operator, then $\Pi(A) \subseteq \Lambda(A)$.

Proof. If $\lambda \notin \Lambda(A)$, then $A - \lambda$ is invertible and consequently we have:

$$11x11 = 11(A - \lambda)^{-1} (A - \lambda)x11 \leq 11A - \lambda 11^{-1} 11Ax - \lambda x11$$

for every vector $x$. This implies that $11Ax - \lambda x11 \geq \varepsilon 11x11$ with $\varepsilon = \frac{11}{11(A - \lambda)^{-1} 11}$ for every vector $x$ and hence that $\lambda \notin \Pi(A)$.

Theorem 29. If $A$ is a normal operator, then $\Pi(A) = \Lambda(A)$.

Proof. In view of theorem 28, it is sufficient to prove that $\Lambda(A) \subseteq \Pi(A)$. If $\lambda \notin \Pi(A)$, then there exists a positive real number $\varepsilon$ such that $11Ay - \lambda y11 \geq \varepsilon 11y11$ for every vector $y$.

Since $A - \lambda$ is just as normal as $A$ and hence $(A - \lambda)^* = A^* - \lambda^*$, it follows from theorem 28 that $11A^*y - \lambda^* y11 \geq \varepsilon 11y11$ for all $y$.

In order to prove that the range of $A - \lambda$ is dense or that the orthogonal complement of the range is $0$. Clearly if a vector $y$ is orthogonal to the range $A - \lambda$ then $0 = ((A - \lambda)x, y) = x, (A^* - \lambda^*)y$ for all $x$ and hence $A^*y - \lambda^* y = 0$ since $11A^*y - \lambda^* y11 \geq \varepsilon 11y11$ it follows that $y = 0$.

$\therefore \Pi(A) = \Lambda(A)$

COMPACTNESS OF SPECTRA:

Theorem 30. If an operator $A$ is such that $11A^* - A11$ then $A$ is invertible.
Proof. Write $111 - \alpha 11 = 1 - \alpha \rho$ where $0 < \alpha \leq 1$

then

$11Ax11 = 11x - (x - Ax)11 \geq 11x11 - 11(1-A)x11$

$\geq 11x11 - (1-\alpha)11x11 = \alpha 11x11$

for every vector $x$. It follows from theorem 16 that it is sufficient in order to prove invertibility of $A$ to show that the range $M$ of $A$ is dense in $H$. The density of $M$ can be proved if $y$ is an arbitrary vector and if $d = \inf \{11y-x11; x \in M\}$, then $d=0$. If $d \geq 0$, then there exists a vector $x$ in $M$ such that $(1-\alpha)11y-x11 < d$. Since $M$ contains both $x$ and $A(y-x)$ and therefore $A^*x + A(y-x)$ it follows that

$d \leq 11y-x11 - A(y-x)11 \leq 111 - A11 \cdot 11y-x11$

$= (1-\alpha)11y-x11 < d$

which is a contradiction.

TRANSFORMS OF SPECTRA:

For instance, $A$ and $B$ are operators, and if $B$ is invertible, it is easy to see that $\Lambda(A) = \Lambda(B^{-1}AB)$

(in view of identity $B^{-1}(A-\lambda)B = B^{-1}AB - \lambda$ the invertibility of the right term is equivalent to the invertibility of $A-\lambda$).

Theorem 31. If an operator $A$ is invertible, then

$\Lambda(A) = (\Lambda(A)^{-1})^{-1}$.

Proof. If $A$ is invertible, then $0$ is not in $\Lambda(A)$, the symbol $(\Lambda(A)^{-1})^{-1}$ makes sense.
The identity \( (A - \lambda I)^{-1} = (A - \lambda I)^{-1} \) shows that if \( \lambda \notin \sigma(A) \) so that \( A - \lambda I \) is invertible, then \( A^* - \lambda I \) is invertible so that
\[
\lambda \notin \sigma(A^*). \] In other words, \( \sigma(A^*) \subseteq (\sigma(A))^* \).

**Spectrum of a Hermitian Operator:**

**Theorem 32.** If \( A \) is a Hermitian operator, then \( \sigma(A) \) is a subset of the real axis.

**Proof.** If \( \lambda \) is not real, then for every non-zero vector \( x \)
\[
0 < |\lambda - \lambda^*|^2 = |(A - \lambda x, x) - (A - \lambda^* x, x)|^2 = |(A - \lambda x, x) - (x, (A - \lambda) x)| \leq 211A x - \lambda x 11 - 11 x 11
\]
the desired follows from the fact that for Hermitian operator the approximate point spectrum and the spectrum are the same.

**Spectral Measure:**

If \( X \) is a set with a specified baloon \( \sigma \)-algebra \( S \) of subsets of spectral measure in \( X \) is a function \( E \) where domains is \( S \) and whose values are idempotent, Hermitian operators (projections) on \( H \), such that \( E(x) = 1 \) and such that
\[
E(U, M) = \sum_{n} E(M) \]. Whenever \( M_n \) is a disjoint sequence of sets in \( S \). A set \( X \) with a specified baloon \( \sigma \)-algebra \( S \) of subsets is usually called a measurable space and is denoted by \( (X, S) \) the sets belonging to \( S \) are called the measurable subsets of \( X \).

**Theorem 33.** If \( E \) is a finitely additive, projection valued set function on the class \( S \) of all measurable subsets of a
measurable space, then $E$ is modular and multiplicative ie if $M$ and $N$ are in $S$, then:

$$E(M \cup N) + E(M \cap N) = E(M) + E(N)$$

and

$$E(M \cap N) = E(M)E(N)$$

$E$ is monotone and subtractive ie if $M$ and $N$ are in $S$ and $M \leq N$, then $E(M) \leq E(N)$ and $E(N - M) = E(N) - E(M)$.

**Proof.** If we add $E(M \cap N)$ to both sides of the equation

$$E(M \cup N) + E(M \cap N) = E(M) + E(N)$$

we obtain

$$E(M \cup N) + E(M \cap N) = (E(M - N) + E(M \cap N))$$

$$+ (E(N - M) + E(M \cap N)) = E(M) + E(N)$$

this proves modularity. Since $E$ is monotone and subtractive $E(M \cap N) \leq E(M \cup N)$ it follows that $E(M)E(M \cap N) = E(M \cap N)$.

**Theorem 34.** A projection valued function $E$ on the class $S$ of measurable subsets of measurable space is a spectral measure if:

(i) $E(x) = 1$

(ii) for each pair of vectors $x$ and $y$, the complex valued set function $M$ defined for every $M$ in $S$ by $M(M) = (E(M)x, y)$ is countably additive.

**Proof.** If $E$ is a spectral measure, then:

(i) holds by the definition
(ii) follows from the fact that inner product are factor of which is an infinite sum may be formed term be term, consequently suppose (i) and (ii) holds. If M and N are disjoint measurable sets, then the identity

$$(E(M \cup N)x,y) = (E(M)x,y)(E(N)x,y)$$

$$((E(M) E(N))x,y)$$

proves that $E(M \cup N) = E(M) + E(N)$ proving that $E$ is finitely additive. If, similarly, $M$ is a disjoint sequence of measurable sets with $\bigcup \nu M_\nu = M$, it is to imply to argue that

$$(E(M)x,y) = \sum \nu (E(M_\nu)x,y) = ((E(M_\nu))x,y)$$

for all $x$ and $y$, and hence that $E(M) = \sum \nu E(M_\nu)$ now the only thing to prove that $\sum \nu E(M_\nu)$ makes sense. The multiplicatively of implies that $[E(M_\nu)]$ is an orthogonal sequence of projectives and hence that $[E(M_\nu)x]$ is an orthogonal sequence of vectors for every $x$.

Since

$$\sum \nu 11E(M_\nu)x11^2 = \sum \nu (E(M_\nu)x,x)$$

$$= (E(M)x,x) = 11E(M)x11^2$$

it follows that the sequence $E(M_\nu)x$ is summable.

If $\sum \nu E(M_\nu)x = Ax$, then it is clear that $A$ is a linear transformation of $H$ into itself.
SPECTRAL INTEGRALS:

In this section we will work with an arbitrary, but fixed measurable space \((X,S)\) the expression "spectral measure" will always refer to spectral measure in \(X\) and also symbol \(B\) will be used for the class of complex valued bounded, measurable functions on \(X\) and with:

\[N(f) = \sup \left\{ \|f\| : \lambda \in X \right\}\] whenever \(f \in B\).

**Theorem 35.** If \(E\) is spectral measure and if \(f \in B\), then there exists a unique operator \(A\) such that \((Ax,y) = \int f(\lambda)d(E(\lambda)x,y)\) for every pair of vectors \(x\) and \(y\), the dependence of \(A\) on \(f\) and \(E\) will be denoted by writing \(A = \int f dE = \int f(\lambda)dE(\lambda)\).

**Proof.** The boundedness of \(f\) implies that the integral \(\Phi(x,y) = \int f(\lambda)d(E(\lambda)x,y)\) may be formed for every pair of vectors \(x\) and \(y\), obviously \(\Phi\) is a bi-linear functional. Since \(\Phi(x,x) = \int f(\lambda)|d(\lambda)|x\|x\|^2\leq N(f)|x||x|^2\) it follows that \(\Phi\) is bounded and hence by theorem 26 there exists a unique operator satisfying the conditions.

**Theorem 36.** If \(E\) is spectral measure, if \(f\) and \(g\) are in \(B\), and if \(\alpha\) is a complex number then

\[\int (\alpha f) dE = \alpha \int f dE,\int (f+g) dE = \int f dE + \int g dE\]

and

\[\int f^* dE = (\int f dE)^*\]

**Proof.** The proofs of all three assertions are similar and
obvious. To prove for instance, the last one we write

\[ A = \int f \, dE \quad \text{and} \quad B = \int f^* \, dE \quad \text{and observe} \]

\[ (x, By) = (By, x)^* = \int f^*(\lambda) dE(\lambda, x) \]

\[ = \int f(\lambda) d(x, E(\lambda)y) = \int f(\lambda) d(E(\lambda)x, y) \]

\[ = (Ax, y) \]

**Theorem 37.** If \( E \) is a spectral measure and if \( f \) and \( g \) are \( B \),

then \( \int fdE \langle \int gdE \rangle = \int gdE \).

**Proof.** We write \( A = \int f \, dE + B = \int g \, dE \). If the measure \( M \) in \( X \) is

defined for every set \( M \) in \( S \) by \( M(f) = (E(M)x, y) \) where \( x \) and \( y \)

are any fixed vectors, then

\[ M(M) = (Bx, E(M)y) = \int g(\lambda) d(E(\lambda)x, E(M)y) \]

\[ = \int g(\lambda) d(E(M)y, E(\lambda)x) \]

\[ = \int g(\lambda) d(E(M \cap \lambda)x, y) \]

\[ = \int g(\lambda) d(E(\lambda)x, y) \]

for every \( M \) in \( S \), it follows that

\[ (ABx, y) = (Ay, Bx)^* = (\int f(\lambda) d(E(\lambda)y, Bx)) \]

\[ = \int f^*(\lambda) d(y, E(\lambda)Bx) \]

\[ = \int f(\lambda) d(E(\lambda)Bx, y) \]

\[ = \int f(\lambda) dM(\lambda) = \int f(\lambda) g(\lambda) d(E(\lambda)x, y) \]

and hence \( AB = \int fg \, dE \).

If \( E \) is a spectral measure and \( B \) is an operator, we shall

write \( E \leftrightarrow B \) for the assertion that \( E(M) \leftrightarrow B \) for all \( M \) in \( S \).
Theorem 39. If $\mathcal{E}$ is a spectral measure, if $B$ is an operator such that $E \leftrightarrow B$ and if $f \in B$, then $\int fd\mathcal{E} \leftrightarrow B$.

Proof. If $\int fd\mathcal{E} = A$, then

$$(ABx, y) = \int f(\lambda) d\mathcal{E}(\lambda) Bx, y)$$

$$= \int f(\lambda) dB \mathcal{E}(\lambda)x, y)$$

$$= \int f(\lambda) d(E(\lambda)x, B^* y) = (Ax, B^* y) = (BAx, y)$$

for every pair of vectors $x$ and $y$. 
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