Special functions of mathematical physics and the solution of their associated differential equations

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SPECIAL FUNCTIONS OF MATHEMATICAL PHYSICS
AND THE SOLUTION OF THEIR ASSOCIATED DIFFERENTIAL EQUATIONS

A THESIS
SUBMITTED TO THE FACULTY OF ATLANTA UNIVERSITY
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OF MASTER OF SCIENCE

BY
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DEPARTMENT OF MATHEMATICS

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CHAPTER I
INTRODUCTION

The modern physical sciences developed from natural philosophy as men realized it was possible to treat motion and time rates in general by mathematical methods. These methods led to "differential equations" as a method of describing phenomena.

The special functions of mathematical physics arise in the solution of partial differential equations governing the behavior of certain physical quantities.

The adjective "special" is used in this connection because here we are not, as in analysis, concerned with the general properties of functions, but only with the properties of functions which arise in the solution of special problems.

These functions are associated with the names of Legendre, Bessel, Hermite and Laguerre. Another related function of importance is the hypergeometric function.

We shall have occasion to discuss ordinary linear differential equations of the second order with variable coefficients whose solutions cannot be obtained in terms of the elementary functions of mathematical analysis. In such cases our procedure is to derive a pair of linearly independent solutions in the form of infinite series. This is the procedure to be followed in these instances. We shall discuss some important properties of infinite series in Chapter II.

A second-order linear differential equation of the form

\[ \frac{d^2w}{ds^2} + p(s) \frac{dw}{ds} + q(s)w = 0, \]
where \( p(z) \) and \( q(z) \) are analytic functions except at a finite number of points will be the object of our investigation. We restrict our attention to this equation for two reasons; firstly, the analysis of this case can easily be extended to the more general case; secondly, the particular functions which we deal with in the sequel do, in fact, satisfy second-order equations. If a solution of this equation exist, we want to know what effect the singularities of \( p(z) \) and \( q(z) \) have on the nature of the solution. This discussion will be presented in our analysis of the methods of Taylor, Picard and Frobenius for finding solutions in Chapter III.

In Chapter IV we shall investigate the development of these special functions and the solution of their associated differential equations.

The importance of these functions lies in the fact that it is often possible to express physical solutions in terms of them.
CHAPTER II

PROPERTIES OF INFINITE SERIES

(This chapter, pp. 3-11, is quoted from Copson1).

"We shall concern our discussion with properties of infinite series which will be useful in establishing the validity of our sequel. Hence, we have the following:

Definition 1. If \( \{ a_n \} \) is a sequence of complex numbers, \( a_0, a_1, \ldots, a_n \ldots \)
we define the symbol \( C_0 + a_1 + a_2 + \ldots + a_n + \ldots \),
which involves the addition of an infinite number of complex numbers, as an infinite series. This symbol has, in itself, no meaning. In order to assign a meaning to the sum of such an infinite series, we consider the associated sequence of partial sums \( S_0, S_1, S_2, \ldots, S_n, \ldots \) where
\[
S_n = a_0 + a_1 + a_2 + \ldots + a_n.
\]
The sequence \( \{ S_n \} \) is the main object of investigation. If this sequence tends to a finite limit \( S \), we say that the infinite series is convergent and that its sum is \( S \); in this case, we write:
\[
\sum_{n=0}^{\infty} a_n = S.
\]
But if the sequence of partial sums does not tend to a finite limit, we say that the infinite series is divergent.

The necessary and sufficient condition for the convergence of an infinite series is provided by Cauchy's principle of convergence, which is stated as the following theorem:

Theorem 1. -- The necessary and sufficient condition for the convergence of the sequence \( \{ S_n \} \), where \( \{ S_n \} \) denotes the sequence of partial sums of the series \( \sum_{n=0}^{\infty} a_n \), is that, given any positive number \( \varepsilon \),

---

there should exist an integer $N$, depending on $\epsilon$, such that the inequality $|S_{n+p} - S_n| < \epsilon$ holds for every positive integer $p$.

The condition is necessary, for if $S_n \to S$, there exists an integer $N$, depending on $\epsilon$, such that the inequality $|S_n - S| < \frac{\epsilon}{2}$ holds when $n \geq N$. Hence, if $p$ is any positive integer, we have

$$|S_{n+p} - S_n| < |S_{n+p} - S_n| + |S_n - S| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

The condition is also sufficient, for if it is satisfied, the given sequence is bounded, since all except a finite number of its points lie in the neighborhood of $S_n$ of radius $\epsilon$; hence, by the Bolzano-Weierstrass theorem, the sequence posses at least one limit point.

The limit point $S$ is unique. For if $S'$ is another limit point of then there exist integers $q$ and $r$ depending on the given number $\epsilon$, such that

$$|S - S_{n+q}| < \frac{\epsilon}{2}, \quad |S' - S_{n+r}| < \frac{\epsilon}{2}.$$

With these values of $n$, $q$, $r$, we have

$$|S - S'| = |(S - S_{n+q}) + (S_{n+q} - S_n) - (S_{n+r} - S_n) - (S' - S_{n+r})|$$

$$\leq |S - S_{n+q}| + |S_{n+q} - S_n| + |S_{n+r} - S_n| + |S' - S_{n+r}|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} + \frac{\epsilon}{2} + \frac{\epsilon}{2} = 2\epsilon$$

Thus $|S - S'| < 2\epsilon$ implies, since $\epsilon$ is arbitrary, that $|S - S'| = 0$, hence, $S = S'$ and the sequence does not possess more than one limit point; this completes the proof of the sufficiency of the condition.

An infinite series $\sum_{n=0}^{\infty} a_n$ is said to be absolutely convergent if the series $\sum_{n=0}^{\infty} |a_n|$ is convergent. We now prove an important theorem.

**Theorem 2.** Absolute convergence of an infinite series implies convergence, but not conversely.
Let us suppose that the series \( \sum_{n=0}^{\infty} a_n \) is absolutely convergent, and let us write

\[
S_n = a_0 + a_1 + a_2 + \cdots + a_n, \quad T_n = |a_0| + |a_1| + |a_2| + \cdots + |a_n|.
\]

Then, given any positive number \( \varepsilon \), such that the inequality

\[
|T_{n+p} - T_n| < \varepsilon
\]

holds for every positive integer \( p \). Now

\[
|S_{n+p} - S_n| = |a_{n+1} + a_{n+2} + \cdots + a_{n+p}| \\
\leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}| \\
= T_{n+p} - T_n,
\]

But since \( T_{n+p} - T_n \) is positive, this gives

\[
|S_{n+p} - S_n| \leq |T_{n+p} - T_n| < \varepsilon
\]

Hence, by Cauchy's principle of convergence, the series \( \sum_{n=0}^{\infty} a_n \) is convergent.

To complete the proof of the theorem, we need only observe that the series

\[
\frac{i}{1} + \frac{i^2}{2} + \frac{i^3}{3} + \cdots + \frac{i^n}{n} + \cdots
\]

is convergent, but not absolutely convergent.

**Cauchy's nth Root Test.** -- The series \( \sum a_n \) of complex terms is absolutely convergent if \( \lim |a_n|^{1/n} \), but is divergent if \( \lim |a_n|^{1/n} \).

For if \( \lim |a_n|^{1/n} = |c| \), where \( 0 < c < \frac{1}{e} \), then the inequality

\[
|a_n|^{1/n} \leq |c|\]

is true for all except a finite number of values of \( n \). There exists, therefore, an integer \( M \) such that \( |a_n| \leq (1 - c)^n \) when \( n \geq M \).
If, now, $T_n$ denotes the nth partial sum of the series $\sum |a_n|$, we have, when $n \geq M$,

$$T_{n+q} - T_n \leq (1 - c)^{n+1} + (1 - c)^{n+2} + \cdots + (1 - c)^{n+q} \leq \frac{(1 - c)^{n+1} - (1 - c)^{n+p+1}}{c} < \frac{(1 - c)^{n+1}}{c}.$$ 

Since this last expression tends to zero as $n \to \infty$, we may assign an arbitrary positive number $\epsilon$ and then choose an integer $N(>M)$ such that the inequality $|T_{n+q} - T_n| < \epsilon$ is true for all positive integers $p$. Hence, by Cauchy's principle of convergence, the series is convergent.

On the other hand, if $\lim |a_n|^{1/n} = |1 + 3d$, where $d > 0$, the inequality $|a_n|^{1/n} \geq |1 + d|$ is true for an infinite number of values of $n$. This, however, implies that $|a_n| \geq (1 + d)^n$ for an infinite number of values of $n$, and so $A_n$ does not tend to zero as $n \to \infty$. Hence, the series $\sum a_n$ is divergent.

**Power Series.** An infinite series, proceeding in ascending integral powers of $z$, of the form

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

where the coefficients $a_0, a_1, a_2, \ldots, a_n$ are all constants, is called a power series. By Cauchy's nth root test, this series converges absolutely when $\lim |a_n z^n| < 1$ and diverges when $\lim |a_n z^n| > 1$. Hence, if $\lim |a_n z^n|^{-1/n} = R$

the series is absolutely convergent when $|z| < R$ and divergent when $|z| > R$. The number $R$ is called the radius of convergence of the power series; the circle $|z| = R$ is its circle of convergence.

There are three cases to be considered; namely, (i) $R = 0$, (ii) $R$ finite, (iii) $R$ infinite. The first case is quite trivial, since the series is then
convergent only when \( Z = D \). In the third case, the series converges for every finite value of \( z \). The second case is to be investigated.

Hence, we shall now show the following theorem:

**Theorem 3.** — If a power series has a non-zero radius of convergence, its sum is an analytic function regular within its circle of convergence.

Let \( f(z) \) be the sum of the power series \( \sum_{n=0}^{\infty} a_n z^n \) which has a non-zero radius of convergence \( R \). Obviously, \( f(z) \) is one-valued when \( |z| < R \); we have to show that it is continuous and differentiable at every point of the domain \( |z| \leq R_1 \), where \( R_1 \) is any finite positive number definitely less than \( R \).

Now choose a number \( R_2 \) such that \( 0 < R_2 < R_1 < R \). If \( |z| \leq R_1 \) and \(|h| \leq R_1 - R_2 \), we have \( |z + h| \leq R_2 \), and so

\[
\begin{align*}
    f(z+h) - f(z) &= \sum_{n=0}^{\infty} a_n (z+h)^n - \sum_{n=0}^{\infty} a_n z^n \\
    &= \sum_{n=1}^{\infty} a_n [(z+h)^n - z^n] \\
    &= \sum_{n=1}^{\infty} a_n \left[ (z+h)^{n-1} + (z+h)^{n-2}z + \ldots + (z+h)z^{n-2} + z^{n-1} \right]
\end{align*}
\]

From this, it follows that

\[
|f(z+h) - f(z)| < |h| \sum_{n=1}^{\infty} |a_n| \frac{R_2^{n-1}}{R_1^{n-1}}.
\]

But this series of positive terms is convergent, since

\[
\lim_{n \to \infty} \left\{ |a_n| \frac{R_2^{n-1}}{R_1^{n-1}} \right\}^{1/n} = \frac{R_2}{R_1} < 1.
\]

Hence,

\[
|f(z+h) - f(z)| < A |h|,
\]

where \( A \) is independent of \( z \) and \( h \), and therefore \( f(z) \) is continuous when \( |z| \leq R_1 \).
Now the increment ratio $\frac{f(z+h)-f(z)}{h}$ tends formally to $\sum_{n=1}^{\infty} a_n z^{n-1}$ as $h \to 0$. Accordingly, we consider the expression

$$I = \frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} a_n z^{n-1}$$

$$= \sum_{n=1}^{\infty} a_n \left\{ \frac{(z+h)^n - Z^n}{h} - n Z^{n-1} \right\},$$

where $z$ and $h$ satisfy the same condition as before. By the use of the binomial theorem, we see that

$$\frac{(z+h)^n - Z^n - Z^{n-1}}{h} = h \sum_{r=0}^{n} \binom{n}{r} C_r Z^{n-r} h^{r-1},$$

and so

$$\left| \frac{(z+h)^n - Z^n - Z^{n-1}}{h} \right| \leq |h| \sum_{r=0}^{n} \binom{n}{r} C_r \left| Z \right|^{n-r} |h|^{r-2}$$

$$= |h| \sum_{r=0}^{n-1} \binom{n}{r} C_{r+1} \left| Z \right|^{n-r-2} |h|^{r}$$

$$\leq \frac{1}{2} n(n-1) |h| \left\{ |Z| + |h| \right\}^{n-2}$$

$$\leq \frac{1}{2} n (n-1) |h| R_1^{n-2}.$$

Hence

$$|I| \leq \frac{1}{2} |h| \sum_{n=1}^{\infty} n(n-1) |a_n| R_1^{n-2} = B |h|,$$

where $B$ is finite and independent of $z$ and $h$, since the series $\sum_{n=1}^{\infty} n(n-1) |a_n| R_1^{n-2}$ is convergent. But this implies that $I$ tends to zero with $h$. We have thus shown that

$$f'(z) = \sum_{n=1}^{\infty} a_n z^{n-1},$$

provided that $|Z| \leq R_1$, where $R_1$ is any number less than the radius of convergence of the given power series. This completes the proof of the theorem.
Now, since \( \lim n A_n^{-\frac{1}{n}} = \lim |A_n|^{-\frac{1}{n}} \) and 
\[
\lim |n A_n|^{-\frac{1}{n}} = \lim |A_n|^{-\frac{1}{n}},
\]
the series \( \sum n A_n Z^{-n} \) has the same radius of convergence \( R \) as the series \( \sum A_n Z^n \) whose sum is \( f(z) \). Hence, if we apply the theorem to the derived series, we see that

is itself an analytic function, regular when \( |Z| < R \), and that its
derivative is \( \sum n(n-1) A_n Z^{-n-2} \) and so on. Thus the derivative of \( f(z) \)
any order \( p \) is regular when \( |Z| < R \), and is given by the formula

\[
\frac{d^p}{dz^p} f(z) = \sum_{n=p}^{\infty} n C_n A_n Z^{-n-p}.
\]

Hence, we have also proved another important theorem as follows:

**Theorem 4.** -- A power series can be differentiated term by term as
often as we please at any point within its circle of convergence.

We now prove the converse to Theorem 4.

**Theorem 5.** -- If \( f(z) \) is an analytic function regular in a neighbor-
hood of \( Z = \alpha \), it is expansible as a power series of the form \( \sum A_n (Z-\alpha)^n \),
whose radius of convergence is not zero.

By hypothesis, there exists a positive number \( R \) with the property
that \( f(z) \) is regular \( |Z-\alpha| < R \). Let \( R_1 \) be any positive number less
than \( R \), and let \( R_2 = \frac{1}{\alpha}(R+R_1) \), so that \( 0 < R_1 < R_2 < R \). Then \( f(z) \)
is certainly regular within and on the circle \( C \) whose equation is

Now let \( \alpha + h \) be any point of the region \( |Z-\alpha| \leq R_1 \). Then as \( \alpha + h \)
is within the circle \( C \), we find, by using Cauchy's integral formula, that

\[
f(\alpha + h) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - (\alpha + h)} \, dz
\]

\[
= \frac{1}{2\pi i} \int_C f(z) \left\{ \frac{1}{z - \alpha} + \frac{h}{(z - \alpha)^2} + \cdots + \frac{h^n}{(z - \alpha)^{n+1}} + \frac{h^{n+1} \alpha}{(z - \alpha)^{n+1}(z - \alpha - h)} \right\} \, dz.
\]
If we now use Cauchy's formula for the derivatives of an analytic function, we obtain

\[ f(a + h) = f(a) + \sum_{r=1}^{n} f^{(r)}(a) \frac{h^r}{r!} + A_n, \]

where

\[ A_n = \frac{h^{n+1}}{2\pi i} \int_{C} \frac{f(z)}{(z-a)^{n+1}} \, dz \]

But \(|f(z)|\) is continuous on the circle \(C\) and so is bounded there. Hence, there exists a positive number \(M\) such that \(|f(z)| \leq M\) on \(C\).

Moreover, when \(|z - a| = R_2\),

\[ |z-a-h| \geq \frac{1}{2} \left( |z-a| + |h| \right) \geq R_2 - R_1. \]

Therefore, we find that

\[ |A_n| \leq \frac{M|h|}{R_2 - R_1} \left( \frac{|h|}{R_2} \right)^n. \]

Since, however, \(|h| \leq R_1 < R_2\), it follows from this inequality that \(A_n\) tends to zero as \(n\) tends to infinity and therefore that

\[ f(a + h) = f(a) + \sum_{n=1}^{\infty} f^{(n)}(a) \frac{h^n}{n!}. \]

It is, however, possible to prove rather more than the mere convergence of this power series. Since \(|h| \leq R_1\), we have

\[ |A_n| \leq \frac{M R_1}{R_2 - R_1} \left( \frac{R_1}{R_2} \right)^n \]

the expression on the right-hand side of the inequality being independent of \(h\). Hence, given any positive number \(\epsilon\), we can choose an integer \(N\),
depending on $\varepsilon$, but quite independent of $h$, such that $|A_n| < \varepsilon$ when $n > N$.

We express this property by saying that the power series converges uniformly with respect to $h$ when $|h| \leq R_1$.

We have now proved the following theorem:

Theorem 6. -- If $f(z)$ is an analytic function regular in the neighborhood as a convergent power series of the form

$$f(z) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

This expansion is uniformly convergent when $|z-a| < R$, provided that $R < R_1$.

This result is known as Taylor's theorem concerning analytic functions of a complex variable.\(^1\)

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\(^1\) Ibid.
CHAPTER III
EXISTENCE THEOREMS AND METHODS OF SOLUTION

(This section, pp. 12-22, is quoted, except for a few minor changes, from Copson\textsuperscript{1}).

"Before we can discuss in detail the properties of several important analytic functions defined by an associated differential equation, it is desirable to consider whether there exists an analytic function which satisfies the homogeneous linear differential equation

\[ \frac{d^n \omega}{dz^n} + \varphi_1(z) \frac{d^{n-1} \omega}{dz^{n-1}} + \cdots + \varphi_n(z) \frac{d \omega}{dz} + \varphi(z) \omega = 0, \]

where the coefficients \( \varphi_1(z), \varphi_2(z), \ldots, \varphi_n(z) \) are analytic functions whose only singularities of finite affix are poles, and, further, if such a solution does exist, to ask what effect the singularities of the coefficient have on the nature of the solution.

The simplest equation of this type \( \frac{d \omega}{dz} + \varphi(z) \omega = 0 \) presents little difficulty. Its variables are separable, and the solution is

\[ \omega = e^{\int \varphi(z) \, dz}. \]

On the other hand, the equation of order two

\[ \frac{d^2 \omega}{dz^2} + \varphi(z) \frac{d \omega}{dz} + \varphi(z) \omega = 0 \]

has no such simple solution. We restrict our attention to this equation for two reasons; firstly, the analysis in this case is easily extended to the general case, and, secondly, the special functions with which we discuss in the following chapter do, in fact, satisfy second-order equations.

\textbf{Definition 2.} \( \omega = Z \) is called an ordinary point of \( \omega'' + \varphi(z) \omega' + \varphi(z) \omega = 0 \) if \( \varphi(z) \) and \( \varphi'(z) \) are regular in a neighborhood of \( Z = Z_0 \). All other

\textsuperscript{1} Ibid., pp. 233-43.
points are called singular points of this equation. The solution near an ordinary point, we now show the following theorems:

**Theorem 6.** If \( z = z_0 \) is an ordinary point of \( w'' + \phi(z)w' + \psi(z)w = 0 \) and if \( C_0 \) and \( C_1 \) are two arbitrary constants, there exists a unique function \( \omega(z) \) which is regular and satisfies the differential equation in a certain neighborhood of \( z = z_0 \), and which also satisfies the initial conditions \( \omega(z_0) = C_0, \omega'(z_0) = C_1 \).

For simplicity, we suppose that \( z_0 = 0 \), then since, \( \phi(z) \) and \( \psi(z) \) are regular in a neighborhood \( |z| < R \) of the origin, they are expansible as Taylor series of the form

\[
\phi(z) = \sum_{n=0}^\infty \phi_n z^n, \quad \psi(z) = \sum_{n=0}^\infty \psi_n z^n,
\]

the radius of convergence of each series being not less than \( R \). We now try to find a formal solution by substituting

\[
\omega = C_0 + C_1 z + C_2 z^2 + \cdots
\]

in the equation

\[
\frac{d^2 \omega}{dz^2} + \sum_{n=0}^\infty \phi_n z^n \frac{d \omega}{dz} + \sum_{n=0}^\infty \psi_n z^n \omega = 0
\]

and equating coefficients. This gives

\[
-2 C_2 = C_1 \phi_0 + C_0 \psi_0, \quad 3 C_3 = 2 C_3 \phi_0 + C_1 \phi_1 + C_0 \psi_1, \quad \text{and, generally,}
\]

\[
-(n-1)n C_n = (n-1)C_{n-1} \phi_0 + (n-2)C_{n-2} \phi_1 + \cdots + C_1 \phi_{n-2} + C_{n-2} \psi_0 + C_{n-3} \psi_1 + \cdots + C_1 \psi_{n-3} + C_0 \psi_{n-2}.
\]
These equations determine the coefficient $A_n$ successively as linear combination of $A_k$ and $A_{k+1}$.

We next show that this power series which satisfies the given initial conditions, has a radius of convergence which is not less than $R$. The proof of this is rather difficult, since we know very little about the coefficient $p_r$ and $q_r$.

Let us denote by $M$ and $M_r$ the maximum values, necessarily finite, of $|f(z)|$ and $|q(z)|$ on the circle $|z| = r$, where $r < R$. Then, by Cauchy's inequality we have

$$|q_n| \leq M/r^n,$$

and so

$$|q_n| \leq K/r^n,$$

where $K$ is the greater of $M$ and $M_r$.

Writing $\ell_0$ and $\ell_1$ for $|A_0|$ and $|A_1|$, respectively, we have

$$2|A_1| \leq \frac{|f_0|}{\ell_0} + \frac{a}{\ell_0} q_0 \leq \ell_1 K + \ell_0 K/r \leq 2 \ell_1 K + \ell_0 K r$$

and so

$$|A_1| \leq \ell_1,$$

where $2 \ell_2 = 2 \ell_1 K + \ell_0 K r/r$.

Similarly,

$$2 \cdot 3 |A_2| \leq 2 |A_2| |f_1| + \ell_1 |q_1| + \ell_0 |q_0| + \ell_0 |q_1|$$

$$\leq 2 \ell_2 K + 2 \ell_1 K r^{-1} + \ell_0 K r^{-2}$$

and so

$$|A_2| \leq \ell_2,$$

where $2 \cdot 3 \ell_3 = 3 \ell_2 K + 2 \ell_1 K r^{-1} + \ell_0 K r^{-2}$.

Proceeding in this way, we find that $|A_n| \leq |\ell_n|$ where

$$(n-1)!^{n-1} \ell_n - (n-2)(n-1)! \ell_{n-1} \gamma^{-1} = n \ell_{n-1} K.$$  

But this gives

$$\frac{\ell_n}{\ell_{n-1}} = \frac{n-2}{n} + \frac{k}{n-1} \to \frac{1}{r} \text{ as } n \to \infty,$$
so that the radius of convergence of the power series $\sum a_n z^n$ is $r$. Since, however, $|a_n| \leq |b_n|$, it follows by the comparison test that the radius of convergence of $\sum a_n z^n$ cannot be less than $r$; moreover, as $r$ was any number less than $R$, this shows that $\sum a_n z^n$ converges when $|z| < R$.

The function

$$\omega(z) = \sum a_n z^n$$

is, therefore, regular when $|z| < R$ and satisfies the prescribed conditions at the origin. The formal processes of term-by-term differentiation, multiplication, and rearrangement of power series by which this function was made to satisfy the differential equation are now seen to be completely justified, since all the series involved converge uniformly and absolutely in every closed domain within $|z| = R$. This completes the proof of the theorem.

Since $a_n$ is a linear function of $a_0$ and $a_1$, we can express the solution we have just found in the form $\omega(z) = A_0 \omega_0(z) + A_1 \omega_1(z)$. The function $\omega_0(z)$ is a solution of the differential equation which satisfies the initial condition $\omega_0(0) = 1$, $\omega'_0(0) = 0$, whereas $\omega_1(z)$ is a solution satisfying the conditions $\omega_1(0) = 0$, $\omega'_1(0) = 1$. Every solution of the differential equation regular in the neighborhood of the origin is, therefore, a linear combination of the solutions of $\omega_0(z)$ and $\omega_1(z)$, which we call a fundamental pair of solutions. Obviously $\omega_0(z)$ and $\omega_1(z)$ are linearly independent; by this we mean that there is no linear combination of them which is identically zero.

So far, the functions $\omega_0(z)$ and $\omega_1(z)$ are defined in a neighborhood of the origin. When we continue these functions analytically, they remain linearly independent solutions of the differential equations. The continuation can be carried out along any path which does not pass through a singular point of the differential equation, and so the solution will ultimately
be defined all over the $z$-plane.

The nature of the solution near a regular singularity. -- The point $z_0$ is a singularity of the differential equation $\omega'' + \psi(z) \omega' + \varphi(z) \omega = 0$ if it is a pole of one or both of the functions $p(z)$ and $q(z)$. We call it a regular singularity if it is not a singularity of either of the functions $(z - z_0) p(z)$ and $(z - z_0)^2 q(z)$; otherwise it is called an irregular singularity.

The reason for this distinction is simply explained. In a neighborhood of a regular singularity, the differential equation possesses two linear independent solutions which are regular except possibly for a branch-point at the singularity.

If the origin is a regular singularity of the differential equation $\omega'' + \psi(z) \omega' + \varphi(z) \omega = 0$ under consideration, the functions $Z - \psi(z)$ and $Z^2 q(z)$ are regular in a neighborhood $|Z| < R$ of the origin and so possess convergent Taylor expansions of the form

$$Z \psi(z) = \sum_0^\infty \psi_r Z^r, \quad Z^2 q(z) = \sum_0^\infty q_r Z^r,$$

where the coefficients $\psi_r$, $q_r$, and $q_j$ are not all zero.

We now show that, in general, the equation possesses two linear independent solutions of the form

$$\omega(z) = Z^d \sum_0^\infty C_r Z^r,$$

where $d$ is a root of a certain quadratic equation. When we substitute these power series in the differential equation and equate coefficients, we find that this expression is a formal solution of the equation if $d$ and the co-
efficient $a_r$ satisfy the conditions
\[ G \cdot F(a) = 0, \]
and
\[ F(a + n) = - \sum_{k=0}^{n-1} G_k \left((a+s)q_{n-k} + q_{n-k-s}\right)^k \quad (n \geq 1), \]
where $F(d)$ denotes the quadratic $\lambda (d-1) + q_0$. This equation is called the indicial equation and its roots the exponents of the regular singularity under consideration. The remaining equations determine successively the coefficients $G_k$ as constant multiples of $G_0$ provided that $F(a + n)$ does not vanish for any positive integral value of $n$. Hence, if the indicial equation has distinct roots which do not differ by an integer, this process gives two formal solutions, corresponding to each root of the indicial equation. If, however, the roots of the indicial equation are equal or differ by an integer, we may obtain only one formal solution.

**Convergence of the series solution near a regular singularity**. We have seen that the differential equation
\[ \frac{d^2 \omega}{dZ^2} + q(Z) \frac{d \omega}{dZ} + q(Z) \omega = 0 \]
always has one formal solution $\omega = Z^d \sum G_n Z^n$ valid near the regular singularity at the origin, and that it has two such solutions when the difference of the exponents is not an integer or zero. To prove that this formal series does show wether that the series $\sum G_n Z^n$ terminates or else that it has a non-zero radius of convergence.

Let us suppose $\sum G_n Z$ does not terminate. We must show that if $Zq(Z)$ and $Z^2q(Z)$ are regular when $|Z| < R$, the radius of convergence of $\sum G_n Z^n$ is not less than $R$. 

If \( \lambda' \) is the other root of the indicial equation, the coefficients are given by

\[
\eta (\eta + \lambda - \lambda') a_n = - \sum_{s=0}^{n-1} b_s \left\{ (\eta + s) \varphi_{n-s} + q_{n-s} \right\}.
\]

Let us write \( \ell_n = |a_n| \) when \( 0 \leq n \leq s \), where \( s = |\lambda - \lambda'| \) and \( \mu = |\lambda| \).

Then if \( m \) is the least integer greater than \( s \), we have

\[
m (m-s) |a_m| \leq m (m + \lambda - \lambda') |a_m|
\]

\[
= \left| \sum_{s=0}^{m-1} b_s \left\{ (s + \mu) |\varphi_{m-s}| + |q_{m-s}| \right\} \right|
\]

\[
\leq \sum_{s=0}^{m-1} b_s \left\{ (s + \mu) |\varphi_{m-s}| + |q_{m-s}| \right\}.
\]

But if \( M \) and \( N \) denote the maximum values, necessarily finite, of \( |Z \varphi (z)| \) and \( |Z^2 \varphi (z)| \) respectively or the circle \( |Z| = R \), where \( r < R \), Cauchy's inequality gives

\[
|\varphi_n| \leq M/r^n, \quad |q_n| \leq N/r^n
\]

and so

\[
|\varphi_n| \leq K/r^n, \quad |q_n| \leq K/r^n
\]

where \( K \) is the greater of \( M \) and \( N \). Substituting these bounds for \( |\varphi_n| \) and \( |q_n| \), we find that \( |a_m| \leq b_m \), where

\[
m (m-s) b_m = K \sum_{s=0}^{m-1} (s + \mu + 1) b_s / r^{m-s}.
\]

Similarly we can show that \( |a_n| \leq b_n \) when \( n \geq m \), where

\[
n (n-s) b_n = K \sum_{s=0}^{n-1} (s + \mu + 1) b_s / r^{n-s}.
\]

From this equation, we see that the coefficient \( b_n \) satisfy the recurrence formula

\[
n (n-s) b_n - (n-1) (n-1-s) b_{n-1} / r = K (n+\mu) b_{n-1} / r
\]

But this gives

\[
\frac{b_n}{b_{n-1}} = \frac{(n-1) (n-1-s)}{n (n-s) r} + \frac{K (n+\mu)}{n (n-s) r}
\]

and so \( \lim \left( b_n / b_{n-1} \right) = 1/r \) as \( n \to \infty \).
The radius of convergence of the series \( \sum c_n z^n \) is therefore \( r \). Since, however, \( |c_n| \leq |c_k| \), it follows by the comparison test that the radius of convergence of \( \sum c_n z^n \) cannot be less than \( r \); moreover, as \( r \) was any number less than \( R \), this implies that this series is convergent when

The formal processes which made \( \omega = \sum a_n z^n \) a solution are now seen to be completely justified since all the series involved converge uniformly and absolutely in every closed domain within \( |z| = R \).

When the exponent-difference is not an integer or zero, we derive in a similar manner a second independent solution \( \omega = z^{d'} \sum a'_n z^n \) corresponding to the other root of the indicial equation. In this case at least one of the solutions has a branch-point at the origin.

The exponent-difference is an integer or zero. -- When \( \alpha - \alpha' = \mu \), where \( \mu \) is a positive integer or zero, the solution of \( \omega'' + f(z)\omega' + q(z)\omega = 0 \) fails. For if \( \mu = 0 \), the two solutions become identical, while if \( \mu > 0 \), all the coefficients in one of the solutions from some point onwards are either infinite or indeterminate. It is, however, well known that a knowledge of one solution of a linear differential equation of order \( \mu \) enables us to depress the order to \( \mu - 1 \). In our case, we obtain in this way a linear equation of the first order which can be integrated immediately. To effect this depression of order, we make, according to the usual rule, the change of independent variable

\[
\omega = \omega_0(z) \psi,
\]

where \( \omega_0(z) \) is the known solution of exponent \( \alpha \). The function \( \psi \) is found
to satisfy the equation
\[ \frac{d^3 v}{dz^3} + \left( \frac{2\omega' \mu}{\omega} + \rho \right) \frac{dv}{dz} = 0, \]
whose solution is
\[ v(z) = A + B \int \frac{1}{\omega(z)^3} e^{\int \frac{z}{\omega(z)} dz} \left\{ - \int \frac{z}{\omega(z)} dz \right\} dz, \]
where \( A \) and \( B \) are arbitrary constants. Hence, the repaired second solution, valid near the origin, is
\[ \omega(z) = \omega_0(z) \int \frac{1}{\omega_0(z)^3} e^{\int \frac{z}{\omega_0(z)} dz} \left\{ - \int \frac{z}{\omega_0(z)} dz \right\} dz. \]

Now \( d \) and \( d - \mu \) are the roots of the indicial equation
\[ d^2 + (\rho_0 - 1) d + \rho_0 = 0, \]
so that \( \rho_0 = 1 - \mu - a \). Hence, we have
\[ \frac{1}{\omega_0(z)^3} e^{\int \frac{z}{\omega_0(z)} dz} \left\{ - \int \frac{z}{\omega_0(z)} dz \right\} dz = \frac{1}{Z^{d+1} - \mu} e^{\int \frac{z}{\omega(z)} dz} \left\{ - \int \frac{z}{\omega(z)} dz \right\} dz = Z^{1-\mu} g(z), \]
where \( g(0) = 1/\alpha_0^3 \). Since \( \alpha_0 \neq 0 \), the function \( (\alpha_0 + \alpha_2 + \ldots)^3 \) is regular in the neighborhood of the origin. Hence \( g(z) \) is also regular there and can be expanded as a convergent Taylor series \( g(z) = \sum g_n z^n \). Substituting this series for \( g(z) \), we find that the general solution is
\[ \omega = \omega_0(z) \int Z^{1-\mu} \sum g_n z^n dz \]
\[ = \omega_0(z) \left\{ \sum_{n=0}^{\infty} \frac{g_n z^{n-\mu}}{n-\mu} + g_\mu \log z + \sum_{n=\mu+1}^{\infty} \frac{g_n z^{n-\mu}}{n-\mu} \right\}. \]

In particular, when the exponent-difference \( \mu \) is zero, this solution can
be written in the form
\[ \omega = g_0 \omega_0(z) \log z + z^{d+1} \sum_{n=0}^{\infty} c_n z^n. \]
As \( g_0 \) is not zero, this solution possesses a branch-point at the origin.
When the exponent-difference \( \mu \) is a positive integer, the second solution takes the form
\[ \omega = g_\mu \omega_0(z) \log z + z^{d'} \sum_{n=0}^{\infty} c_n z^n. \]
If it happens, as may be the case, that \( g_\mu \) is zero, the second solution does not involve a logarithmic term.

Solutions which for large values of \(|z|\). -- To discuss the nature of the solution in the neighborhood of the point at infinity, we make the transformation \( z = 1/t \). The differential equation
\[ \frac{d^2 \omega}{dz^2} + \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \omega \right) + q(z) \omega = 0 \]
then becomes
\[ \frac{d^3 \omega}{dz^3} + \left\{ -\frac{2}{t^2} - \frac{1}{t^3} q(1/t) \right\} \frac{d\omega}{dz} + \frac{1}{t^4} q(1/t) \omega = 0. \]
The behavior of the solution for large values of \(|z|\) is now determined by solving the transformed equation in the neighborhood of the origin.

Accordingly, we may that the joint at the infinity is an ordinary point if
\[ \frac{2}{t} - \frac{1}{t^2} q(1/t), \quad \frac{1}{t^4} q(1/t), \]
are regular at infinity. The complete solution of the equation, valid in a neighborhood of the point at infinity, is in this case of the form
\[ C_0 + \frac{C_1}{z} + \frac{C_2}{z^3} + \cdots, \]
where \( C_0, C_1 \) are arbitrary constants.

Again, the point \( t = 0 \) is a regular singularity of the transformed equation.
If \( \varphi(1/t) \) and \( \varphi'(1/t) \) are regular there, we say, therefore, that the point at infinity is a regular singularity if \( Z \varphi(Z) \) and \( Z^2 \varphi(Z) \) are regular there. In this case, \( \varphi(Z) \) and \( \varphi(Z) \) are expandible, by Laurent's theorem, in the series of the form

\[
\varphi(Z) = \frac{a_0}{Z^2} + \frac{a_1}{Z} + \frac{a_2}{Z^2} + \cdots,
\]

\[
\varphi'(Z) = \frac{a_0}{Z^2} + \frac{a_1}{Z} + \frac{a_2}{Z^2} + \cdots,
\]

convergent in a neighborhood \( |Z| > R \) of the point at infinity.

It may now be shown that there exist in this neighborhood two linear independent solutions

\[
\omega = Z^{-d} \sum_{n=0}^{\infty} A_n Z^{-n}, \quad \omega' = Z^{-d'} \sum_{n=0}^{\infty} A'_n Z^{-n}
\]

where \( d \) and \( d' \) are the roots of the indicial equation

\[ d^2 - (q_0 - 1) d + q_0 = 0, \]

provided that these roots do not differ by an integer or zero.\(^1\)

The method of Taylor series. --- If \( \omega(Z) \) possesses a Taylor series expansion in a neighborhood \( |Z - Z_0| < R \), where \( R \) is a finite positive number, then,

\[
\omega(Z) = \omega(Z_0) + \omega'(Z_0)(Z - Z_0) + \frac{\omega''(Z_0)(Z - Z_0)^2}{2!} + \cdots
\]

This enables us to write the solution \( \omega(Z) \) to a given differential equation when we can differentiate as often as we please the given differential equation in the neighborhood \( |Z - Z_0| < R \).

We have have seen from theorem 6, with \( Z_0 = 0 \), a solution of this type exists. Now we show Taylor's method by an illustrative example.

We wish to find the solution of the differential equation \( \omega' = Z + \omega + \cdots \).
We assume that $Z = o$ (for which the Taylor series becomes a Maclaurin series). From the differential equation we find

$$
\omega' = Z + c\omega + 1, \quad \omega'' = 1 + \omega', \quad \omega''' = \omega', \quad \omega^{(n)} = \omega^{(n-1)},
$$

assuming $\omega = c$ when $Z = 0$, we find that

$$
\omega'(0) = c + 1, \quad \omega''(0) = c + 2, \quad \omega'''(0) = c + 2, \ldots
$$

Now substituting in Taylor's series expansion, we have

$$
\omega(z) = c + (c + 1)z + \frac{(c + 2)z^2}{2!} + \frac{(c + 3)z^3}{3!} + \ldots
$$

which may be written

$$
\omega(z) = c + (c + 1)z + (c + 2) \left( \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \right)
$$

$$
= (c + 2)e^z - z - 2.
$$

Picard's method of iteration. — (This section, pp. 23-26, is quoted from Murray and Miller\(^1\)). The differential equation

$$
\frac{dy}{d\chi} = f(\chi, y),
$$

where $y = \ell$ when $\chi = a$, can be written

$$
y = \ell + \int_a^\chi f(\chi, y) \, d\chi. \quad (1)
$$

The integral on the right cannot be performed since we do not know $y$ depends on $\chi$. In fact, that is what we are looking for. The method of Picard consists in assuming that: $y = \ell$ is a first approximation to $y$. We denote this first approximation by $y_1 = \ell$. When this value is substituted for $y$ in the integrand of $y_1$, we denote the resulting value of $y$ by $y_2$, which is a second approximation to $y$, that is to say,

$$
\begin{align*}
y_2 &= \ell + \int_a^\chi f(\chi, y_1) \, d\chi \vphantom{\int_a^\chi f(\chi, y_2) \, d\chi}, \\
y_3 &= \ell + \int_a^\chi f(\chi, y_2) \, d\chi \\
&\vdots \\
y_n &= \ell + \int_a^\chi f(\chi, y_{n-1}) \, d\chi.
\end{align*}
$$

The sequence of functions \( \{ y_n \} \) tends to a limit \( y \) as \( n \to \infty \) whenever \( f(x, y) \) obeys certain conditions.

These conditions are that, after suitable choice of the positive numbers \( h \) and \( k \), we can assert that, for all values of \( x \) between \( a - h \) and \( a + h \), and for all values of \( y \) between \( c - k \) and \( c + k \), we find positive numbers \( M \) and \( N \) so that

\[
(i) \quad |f(x, y)| < M \\
(ii) \quad |f(x, y) - f(x, y')| < N|y - y'| \quad , y \text{ and } y' \text{ being any two values of } y \text{ in the range considered.}
\]

We shall now prove that whenever \( f(x, y) \) satisfies (i) and (ii) the sequence \( \{ y_n \} \) tends to a limit. We consider the difference between the successive approximations.

\[
y_1 - \varepsilon = \int_a^x f(x, \varepsilon) \, dx , \quad \text{by definition,}
\]

but

\[
|f(x, \varepsilon)| < M \quad , \quad \text{by condition (i),}
\]

so

\[
|y_1 - \varepsilon| < \left| \int_a^x M \, dx \right| , \quad \text{that is to say} \quad < M|a - a| < M h \quad ... (1)
\]

Also

\[
y_n - y_1 = \varepsilon + \int_a^x f(x, y_1) \, dx - \varepsilon - \int_a^x f(x, \varepsilon) \, dx , \quad \text{by definition,}
\]

\[
= \int_a^x \{ f(x, y_1) - f(x, \varepsilon) \} \, dx ;
\]

but

\[
|f(x, y_1) - f(x, \varepsilon)| < N|y_1 - \varepsilon| , \quad \text{by condition (ii),}
\]

from (1),

\[
< MN|a - a| ,
\]
so \[ |y_n - y_1| < \left| \int_a^x MN (x - a) \, dx \right|, \]
\[ < \frac{1}{2} MN (x - a)^2 < \frac{1}{2} MN h^2 \quad \cdots \cdots \quad (2) \]

Similarly, \[ |y_n - y_{n-1}| < \frac{1}{n!} MN^{n-1} h^n \quad \cdots \cdots \quad (3) \]

Now the infinite series
\[ -\ell + MN h + \frac{1}{2} MN h^2 + \cdots + \frac{1}{n!} MN^{n-1} h^n + \cdots = \frac{M}{N} (e^{Nh} - 1) + \ell \]

is convergent for all values of \( h, N, \) and \( M. \)

Therefore the infinite series
\[ -\ell + (y_1 - \ell) + (y_2 - y_1) + \cdots + (y_n - y_{n-1}) + \cdots \]
each term of which is equal or less in absolute value than the corresponding term of the preceding, is still more convergent.

That is to say that the sequence
\[ y_1 = -\ell + (y_1 - \ell) \]
\[ y_2 = -\ell + (y_1 - \ell) + (y_2 - y_1), \]
and so on, tends to a definite limit, say \( Y(x), \) which is what we wanted to prove.

We must now prove that \( Y(x) \) satisfies the differential equation. The inequalities (1), (2), (3) proves that the series is uniformly convergent. If, then, \( \zeta(x,y) \) is continuous \( y_1, y_2, \ldots, y_n, \ldots \) are continuous also and \( Y \) is a uniformly convergent series of continuous functions; that is \( Y \) is itself continuous, and \( Y - y_{n-1} \), tends uniformly to zero as \( n \) increases.

Hence, from condition (ii), \( \zeta(x,y) - \zeta(x,y_{n-1}) \) tends uniformly to zero.

From this we deduce that
\[ \int_a^x \left\{ \zeta(x,y) - \zeta(x,y_{n-1}) \right\} \]
tends to zero

Thus the limit of the relation \( y_n = -\ell + \int_a^x \zeta(x,y_{n-1}) \, dx \)
is \[ Y = -\ell + \int_a^x \zeta(x,y) \, dx, \]
therefore
\[ \frac{dY}{dX} = f(X, Y), \] and \( Y = \ell \) when \( X = 0 \).

This completes the proof.

We now show by example how Picard's method is used.

Solve \( y' = x + y + 1 \). Assume \( y = \ell \) when \( x = 0 \).

Now by integrating
\[ dy = (x + y + 1) \, dx, \]
we get
\[ y = \ell + \int_0^x (x + y + 1) \, dx. \]

Hence, by theorem our first approximation is
\[ y_1 = \ell \]
and
\[ y_2 = \ell + \int_0^x (x + y_1 + 1) \, dx = \ell + \int_0^x (x + \ell + 1) \, dx = \ell + \frac{x^2}{2} + x \]
\[ y_3 = \ell + \int_0^x (x + y_2 + 1) \, dx = \ell + (\ell + 1) x + \frac{(\ell + 2)}{2!} x^2 + \frac{x^3}{3!} \]

and, in general,
\[ y_n = \ell + (\ell + 1) x + \frac{(\ell + 2)}{2!} x^2 + \ldots + \frac{(\ell + 2)(n-1)}{(n-1)!} x^{n-1} + \frac{x^n}{n!} \]

In its limiting case we see that
\[ \lim_{n \to \infty} y_n = \ell + (\ell + 1) x + \frac{(\ell + 2)}{2!} x^2 + \frac{(\ell + 2)x^3}{3!} + \ldots \]

The method of Frobenius. — In the attempt to discover a series solution of a given differential equation, one may find that solutions of the type
\[ \omega = \sum c_n z^n \]
does not exist.

\[ \text{Ibid.} \]
which is a generalization of \( \sum \alpha_n Z^{n+c} \), when \( c = 0 \).

A series \( \sum \alpha_n Z^{n+c} \) is called a Frobenius type series.

It is useful in finding solutions of

\[
\frac{d^2\omega}{dZ^2} + \varphi(Z) \frac{d\omega}{dZ} + \psi(Z) \omega = 0,
\]

where \( \varphi(Z) \) and \( \psi(Z) \) are analytic functions.

The values of \( C \) are determined by a quadrature equation called the indicial equation. The nature of its roots affect the solution of the differential equation.

There are several different cases to consider.

(i) If the indicial roots differ by a constant which is not an integer or zero, the general solution is always obtained.

(ii) If the indicial roots differ by an integer not equal to zero, then the general solution is obtained by use of the smaller indicial root. It is also possible that no solution is obtained by using the smaller indicial root. However, in all cases a solution can be determined by using the larger root.

(iii) If the indicial roots differ by zero, only one solution is obtained.

We shall now show the different cases concerning the roots of the indicial equation by use of examples.

Case (i)

We wish to find the general solution of \( 4Z\omega'' + 2\omega' + \omega = 0 \).

Assume that \( \omega = \sum \alpha_n Z^{n+c} \), where we define \( \alpha_n = 0 \), \( n < 0 \).

Differentiating,

\[
\omega' = \sum (n+c) \alpha_n Z^{n+c-1}
\]

\[
\omega'' = \sum (n+c)(n+c-1) \alpha_n Z^{n+c-2}
\]
Hence, 
\[ 4Z \omega'' + 2 \omega' + \omega = \sum \left[ 4(n+c)(n+c+1)C_{n+1} + 2(n+c+1)C_n + C_n \right] Z^{n+c} = 0 \]
This implies that 
\[ 4(n+c+1)(n+c)C_{n+1} + 2(n+c+1)C_n + C_n = 0 \]
or 
\[ 2(n+c+1)(3n+2c+1)C_{n+1} + C_n = 0 \quad \text{for all } n. \quad \ldots \quad (1) \]
Putting \( n = -1 \), we have, since \( a_{-1} = 0 \), \( 2c(2c-1)C_0 = 0 \)
Thus \( c = 0, \frac{1}{2} \), since \( C_0 \neq 0 \).

For \( c = 0 \), we have, putting \( c = 0 \) in \( (1) \),
\[ C_{n+1} = \frac{-C_n}{(2n+2)(2n+1)} \quad \text{(recursion formula)} \]
Putting \( n = 0, 1, 2, 3, \ldots \), we readily find
\[ C_1 = -\frac{C_0}{2}, \quad C_2 = \frac{C_0}{2 \cdot 3 \cdot 4} = \frac{C_0}{4!}, \]
\[ C_3 = -\frac{C_0}{5 \cdot 6} = -\frac{C_0}{4! \cdot 5 \cdot 6} = -\frac{C_0}{6!}, \ldots \]
Hence, a solution is
\[ \omega = \sum C_n Z^{n+c} = C_0 \left( 1 - \frac{Z}{2!} + \frac{Z^2}{4!} - \frac{Z^3}{6!} + \cdots \right) \]
or
\[ \omega = A \left( 1 - \frac{Z}{2!} + \frac{Z^2}{4!} - \frac{Z^3}{6!} + \cdots \right) \]
where we replace \( C_0 \) by \( A \) for the arbitrary constant.

For \( c = \frac{1}{2} \), we have from \( (1) \)
\[ C_{n+1} = \frac{-C_n}{(2n+3)(2n+2)} \quad \text{(recursion formula)} \]
Putting \( n = 0, 1, 2, \ldots \), we find
\[ C_1 = -\frac{C_0}{2 \cdot 3} = -\frac{C_0}{3!}, \quad C_2 = -\frac{C_0}{4 \cdot 5} = \frac{C_0}{5!}, \]
Therefore, a second solution is

\[ \omega = \sum C_n Z^{n+c} = C_0 Z^{1/2} \left(1 - \frac{Z}{3!} + \frac{Z^2}{5!} - \frac{Z^3}{7!} + \cdots \right) \]

or

\[ \omega = B \left(Z^{1/2} - \frac{Z^{3/2}}{3!} + \frac{Z^{5/2}}{5!} - \frac{Z^{7/2}}{7!} + \cdots \right) \]

where \( B \) is replacing \( C_0 \) for the arbitrary constant.

The general solution is thus:

\[ \omega = A \left(1 - \frac{Z}{2!} + \frac{Z^2}{4!} - \frac{Z^3}{6!} + \cdots \right) + B \left(Z^{1/2} - \frac{Z^{3/2}}{3!} + \frac{Z^{5/2}}{5!} - \frac{Z^{7/2}}{7!} + \cdots \right) \]

The general solution can be written in terms of elementary functions as

\[ \omega = A \cos Z^{1/2} + B \sin Z^{1/2}. \]

**Case (ii)**

We wish to find the general solution of

\[ \omega'' + 2 \omega' + \omega = 0 \]

We proceed as before, in Case (i). Substituting \( \omega = \sum C_n Z^{n+c} \) into the given differential equation we find,

\[ \omega'' + 2 \omega' + \omega = \sum [(n+c+1)(n+c+2)C_{n+2} + (n+c)C_n + C_n] Z^{n+c} = 0 \]

This implies that

\[ (n+c+1)(n+c+2)C_{n+2} + (n+c)C_n + C_n = 0 \quad \cdots \quad (i) \]

putting \( n = -2 \), we have, since \( A_{-2} = 0 \), \( c (c-1) A_0 = 0 \).

Thus \( c = 0, 1 \), since \( A_0 \neq 0 \).

For \( c = 0 \), from (i), we have

\[ C_{n+2} = \frac{-C_n}{(n+2)} \]
putting \( n = 0, 1, 2, \ldots \) we find that
\[
\begin{align*}
A_0 &= -\frac{Q_0}{a}, \quad A_1 = -\frac{Q_1}{3}, \quad A_2 = \frac{Q_2}{2.4}, \\
A_3 &= \frac{Q_3}{3.6}, \quad A_4 = \frac{Q_4}{3.8}, \quad \ldots
\end{align*}
\]
Hence, a solution is
\[
\omega = \sum A_n Z^{n+c} = \left( A_0 + A_1 Z - \frac{A_2 Z^2}{3} - \frac{A_3 Z^3}{2.4} - \frac{A_4 Z^4}{3.6} - \cdots \right)
\]
\[
= A_0 \left( 1 - \frac{Z^2}{2} + \frac{Z^4}{2.4} - \cdots \right) + A_1 \left( Z - \frac{Z^3}{3} + \frac{Z^5}{3.6} - \cdots \right)
\]
or
\[
\omega = A \left( 1 - \frac{Z^2}{2} + \frac{Z^4}{2.4} - \cdots \right) + B \left( Z - \frac{Z^3}{3} + \frac{Z^5}{3.6} - \cdots \right)
\]
where we replace \( A_0 \) and \( A_1 \) by \( A \) and \( B \) respectively.

We have shown that the general solution could be obtained by use of the smaller indicial root.

Case (iii)

Find the general solution of
\[
Z \omega'' + \omega' + Z \omega = 0
\]

Proceeding as before, in the previous examples, we have by substitution of
\[
\omega = \sum A_n Z^{n+c}
\]
into the given differential equation,
\[
Z \omega'' + \omega' + Z \omega = \sum \left[ A_{n-1} + (n + c + 1)^2 A_{n+1} \right] Z^{n+c}
\]
Hence,
\[
A_{n-1} + (n + c + 1)^2 A_{n+1} = 0 \quad \ldots \quad - (3)
\]
putting \( n = -1 \), we get, since \( A_{-2} = 0 \), \( c^2 A_1 = 0 \)

Thus \( c = 0, \infty \), since \( A_0 \neq 0 \)

From (3), when \( c = 0 \), we get
Putting \( n = 0, 1, 2, \ldots \), we have
\[
\begin{align*}
C_1 &= 0, \\
C_2 &= -\frac{C_0}{2a^2}, \\
C_3 &= 0, \\
C_4 &= -\frac{C_2}{4a} = -\frac{C_0}{2^3 4a^2}, \\
& \quad \vdots
\end{align*}
\]

Hence, one solution is
\[
\omega = \sum C_n Z^{n+c} = C_0 \left(1 - \frac{Z^2}{2a^2} + \frac{Z^4}{2^3 4a^2} - \frac{Z^6}{2^5 4^2 6a^2} + \cdots \right)
\]
or
\[
\omega = A \left(1 - \frac{Z^2}{2a^2} + \frac{Z^4}{2^3 4a^2} - \frac{Z^6}{2^5 4^2 6a^2} + \cdots \right)
\]
which can be written as \( \omega = A J_0(z) \), where \( J_0(z) \) is called Bessel's function of order zero.

To find the other solution, we will assume \( c \neq 0 \)

Then
\[
C_{n+1} = -\frac{C_{n-1}}{(n+c+1)^2}, \quad n \geq 0
\]
putting \( n = 0, 1, 2, \ldots \), we have
\[
\begin{align*}
C_1 &= 0, \\
C_2 &= -\frac{C_0}{(c+a)^2}, \\
C_3 &= 0, \\
C_4 &= -\frac{C_2}{(c+4)^2} = -\frac{C_0}{(c+2)^2 (c+4)^2}, \\
& \quad \vdots
\end{align*}
\]
and, in general
\[
C_{2n-1} = 0, \quad C_{2n} = \frac{(-1)^n C_0}{(c+2)^2 (c+4)^2 \cdots (c+2n)^2}
\]

Consider now
\[
\Upsilon = \sum C_n Z^{n+c} = C_0 Z^c \left[1 - \frac{Z^2}{(c+a)^2} + \frac{Z^4}{(c+2)^2 (c+4)^2} - \cdots \right]
\]

By direct substitution
\[ Z Y'' + Y' + Z Y = C^2 a^2 Z^{-1} a_0. \]

Differential with respect to \( C \), we get
\[ \frac{\partial}{\partial C} (Z Y'') + \frac{\partial}{\partial C} (Y') + \frac{\partial}{\partial C} (Z Y) = 2 C a^2 a_0 Z^{-1} a + C^2 a, Z^{-1} a_0 Z. \]

It follows that \( \frac{\partial Y}{\partial C} \) at \( C = 0 \) is a solution.

and
\[
\frac{\partial Y}{\partial C} \bigg|_{C=0} = -\frac{Z^2}{2^2} - \frac{Z^4}{2^{3.4^2}} (1 + \frac{1}{2}) + \frac{Z^6}{2^{3.4^2.6^2}} (1 + \frac{1}{3} + \frac{1}{3}) - \cdots
\]

\[ + \log Z \left( 1 - \frac{Z^2}{2^2} + \frac{Z^4}{2^{3.4^2}} - \cdots \right) \]

is a solution.

Therefore, the general solution is
\[
\omega = A J_0 (Z) + B \left[ J_1 (Z) \log Z + \frac{Z^2}{2^2} - \frac{Z^4}{2^{3.4^2}} (1 + \frac{1}{2}) + \frac{Z^6}{2^{3.4^2.6^2}} (1 + \frac{1}{3} + \frac{1}{3}) - \cdots \right]
\]

**Conditions for existence of a Frobenius type solution.** Before we present sufficient conditions for the existence of a Frobenius type series solution of the differential equation
\[
\frac{d^2 \omega}{dZ^2} + \varphi(Z) \frac{d \omega}{dZ} + q(Z) \omega = 0
\]
we need some definitions.

**Definition 3.** -- \( Z = Z_0 \) is an ordinary point of \( \omega'' + \varphi(Z) \omega' + q(Z) \omega = 0 \)

if \( \varphi(Z) \) and \( q(Z) \) are analytic at \( Z = Z_0 \).

**Definition 4.** -- \( Z = Z_0 \) is a singular point of \( \omega'' + \varphi(Z) \omega' + q(Z) \omega = 0 \)

if \( \varphi(Z) \) and \( q(Z) \) are analytic at \( Z = Z_0 \).

**Definition 5.** -- \( Z = Z_0 \) is a regular singular point of \( \omega'' + \varphi(Z) \omega' + q(Z) \omega = 0 \)

if

1. \( Z_0 \) is not an ordinary point.
2. \( (Z - Z_0) \varphi(Z) \) is analytic at \( Z = Z_0 \).
(iii) \((Z-Z_o)^n q(Z)\) is analytic at \(Z = Z_o\).

We have already proved previously that if \(Z = Z_o\) is an ordinary point or regular singular point of the differential equation

\[
\omega'' + \Phi(Z) \omega' + \Psi(Z) \omega = 0
\]

then a solution of the form

\[
\omega = (Z-Z_o)^k \sum_{n=0}^{\infty} a_n (Z-Z_o)^n
\]

always exists. This is called a Frobenius series about \(Z = Z_o\).

When a Frobenius type series solution exists for the differential equation \(\omega'' + \Phi(Z) \omega' + \Psi(Z) \omega = 0\) it will converge:

(i) For all values of \(Z\) if \(Z = Z_o\) is an ordinary point and no finite singular points exist.

(ii) For all values of \(Z\) if \(Z = Z_o\) is a regular singular point and no other finite singular point exists.

(iii) Within the neighborhood \(|Z-Z_o| < R\) is \(Z = Z_o\) is an ordinary point and where \(R\) is the distance to the nearest singularity.

From this information, we can predict the radius of convergence of the Frobenius type series.
CHAPTER IV

SPECIAL FUNCTIONS AND ASSOCIATED DIFFERENTIAL EQUATIONS

The Hypergeometric Function

The hypergeometric function is defined by means of the hypergeometric series
\[ F (a, b; c ; Z) = 1 + \frac{d^1}{d^1 c} Z + \frac{d^2 d + (d + 1) c (c + 1)}{1 \cdot 2 \cdot c (c + 1)} Z^2 + \cdots \]
which is absolutely convergent if \(|Z| < 1\). If \(|Z| = 1\), it converges absolutely if \(c - a - b > 0\).

The point at infinity and the hypergeometric function. -- (This section, pp. 34-37, is quoted from Lass\(^1\)). "To determine whether the point at infinity is an ordinary point or regular singular point of the differential we let \(Z = 1/t\) in the differential equation \(\omega'' + \varphi(z) \omega' + \psi(z) \omega = 0\) and investigate the point \(t = 0\). we have
\[
\frac{d\omega}{dZ} = \frac{d\omega}{dt} \frac{dt}{dZ} = -t^2 \frac{d\omega}{dt}, \quad \frac{d^2\omega}{dZ^2} = t^4 \frac{d^2\omega}{dt^2} + 2t^3 \frac{d\omega}{dt},
\]
so that \(\omega'' + \varphi(z) \omega' + \psi(z) \omega = 0\) becomes
\[
\frac{d^2\omega}{dt^2} + \frac{2t - \varphi(1/t)}{t^2} \frac{d\omega}{dt} + \frac{1}{t^4} \psi(1/t) \omega = 0.
\]
Hence, \(Z = \infty\) is an ordinary point if
(i) \(\frac{2 - \varphi(1/t)}{t^2}\) is analytic at \(t = 0\)
(ii) \(\frac{\varphi(1/t)}{t^4}\) is analytic at \(t = 0\)
for this case a solution can be found in the form
\[
\omega = C_0 + \frac{C_1}{Z} + \frac{C_2}{Z^2} + \frac{C_3}{Z^3} + \cdots
\]
Again, \(Z = \infty\) is a regular singular point if \(Z = \infty\) is not an ordinary point and

if

(iii) \( \frac{2\zeta - \varphi(1/\zeta)}{\zeta} \) is analytic at \( \zeta = 0 \)
(iv) \( \frac{1}{\zeta^2} \varphi(1/\zeta) \) is analytic at \( \zeta = 0 \)

This implies

\[
\frac{1}{\zeta} \varphi(1/\zeta) = \varphi_0 + \varphi_1 \zeta + \varphi_2 \zeta^2 + \ldots
\]

\[
\frac{1}{\zeta^2} \varphi(1/\zeta) = \varphi_0 + \varphi_1 \zeta + \varphi_2 \zeta^2 + \ldots
\]

so that

\[
\varphi(Z) = \frac{\varphi_0}{Z} + \frac{\varphi_1}{Z^2} + \frac{\varphi_2}{Z^3} + \ldots + \frac{\varphi_n}{Z^{n+1}} + \ldots
\]

\[
\varphi(Z) = \frac{\varphi_0}{Z^2} + \frac{\varphi_1}{Z^3} + \frac{\varphi_2}{Z^4} + \ldots + \frac{\varphi_n}{Z^{n+2}} + \ldots
\]

If, moreover, we desire that the origin also be a regular singular point, we must have

\[
\varphi(Z) = \frac{\varphi_0}{Z}, \quad \varphi(Z) = \frac{\varphi_0}{Z^2}.
\]

We consider now the following problem: We look for the differential equation which has exactly three regular singular points at \( Z = 0, 1, \infty \); all other points are to be ordinary points. Since \( Z = \infty \) has at most simple poles at \( Z = 0, Z = 1 \), we know that \( Z(1-Z) \) is an entire function. Hence

\[
Z(1-Z) \varphi(Z) = \sum_{n=0}^{\infty} C_n Z^n
\]

for all \( \). Since \( Z = \infty \) is to be a regular singular point, condition (iii) must be upheld. From (1)

\[
\varphi(1/\zeta) = \frac{\xi^2 \sum \xi \varphi_n(1/\xi)^n}{\xi - 1}
\]

and

\[
\frac{1}{\zeta} \varphi(1/\zeta) = \frac{\xi^2 \sum \xi \varphi_n(1/\xi)^n}{\xi - 1}
\]

must be analytic at \( \zeta = 0 \). This is possible if and only if \( C_n = 0, \quad n > 1 \)

Thus

\[
Z(1-Z) \varphi(Z) = C_0 + C_1 Z
\]

and

\[
\varphi(Z) = \frac{C}{Z} + \frac{A}{1-Z}
\]

With the same type of reasoning, we find that
\( q(z) = \frac{8(z-\omega)(z-\nu)}{z^2(z-1)^2} \)

Let the roots of the indicial equation at \( z=0 \) be \( \alpha = \omega, \beta = \xi \). We also impose the condition that the indicial equations at \( z=0 \) and \( z=1 \) have at least one root equal to zero.

Now at \( z=0 \) we have

\[
q_0 = \lim_{z \to 0} z q(z) = C
\]

also \( q_1 = \lim_{z \to 0} z^2 q(z) = \mu \nu B \)

The indicial equation at \( z=0 \) is

\[
\lambda (\lambda - 1) + \alpha \lambda + \mu \nu B = 0
\]

If \( \alpha = 0 \) is to be a root of this quadratic, we must have \( \mu \nu B = 0 \) so that we pick \( \mu = 0 \) if we desire \( q(z) \neq 0 \). The other root is \( \lambda = 1 - \omega \). If one of the roots of the indicial equation at \( z=1 \) is zero, the other is \( \lambda = 1 + A \) and

\[
q(1) = \frac{B}{Z(Z-1)}
\]

At \( z=\infty \) we have

\[
\frac{z^2 - \varphi(1/\xi)}{\xi^2} = \frac{z^2 - c \xi + A / (\xi - 1)}{\xi^2} = \frac{2 - c - A / (\xi - 1)}{\xi}
\]

and

\[
q_0 = \lim_{\xi \to 1} \xi \frac{z^2 - \varphi(1/\xi)}{\zeta^2} = 2 - c + A
\]

Similarly, \( q_0 = B \), and the indicial equation at \( z=\infty \) is

\[
\lambda (\lambda - 1) + (2 - c + A) \lambda + B = 0
\]

or

\[
\lambda^2 + (1 - c + A) \lambda + B = 0
\]

If the roots of this equation are \( a \) and \( b \), we must have

\[
a + b = -1 + c - A
\]

\[
a b = B, \quad -A = a + b + 1 - c
\]

Our differential equation is

\[
\frac{d^2 \omega}{dT^2} + \left( \frac{c}{Z} + \frac{(1 - c + a + b)}{Z - 1} \right) \frac{d \omega}{dZ} + \frac{a b}{Z(Z-1)} \omega = 0
\]

This is the famous hypergeometric differential equation. Its solution can be written in the form
\[ \omega(z) = \left\{ \begin{array}{c} 0, \infty, z, 1 \\frac{z}{1-c} \, e^{-a-\varepsilon} \end{array} \right\} \]

The top row denotes the regular singular points, and the other rows contain the roots of the indicial equations at each point (singular).

The operator \( \Theta = \frac{z \, d}{dz} \) can be used in solving this differential equation by series method.\(^1\)

**Legendre Functions**

In the discussion of the hypergeometric function and its associated differential equation the regular singular points were distinct. It is useful to consider what happen when two or more regular singular points approach coincidence, a process usually to be as the "confluence" of singularities.

Consider the hypergeometric function

\[ F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{a(a+1) \cdots (a+n-1) \, b(b+1) \cdots (b+n-1)}{c(c+1) \cdots (c+n-1)} \, \frac{z^n}{n!} \]

Now, if we adopt the solution

\[ a(a+1)(a+2) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \]

where \( \Gamma \) is the gamma function.

We can now write \( F(a, b; c; z) \) as

\[ F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)} \sum \frac{\Gamma(a+n) \Gamma(b+n)}{\Gamma(c+n)} \, \frac{z^n}{n!} \]

Now, let \( z' = b \, z \), and then replace \( z' \) by \( z \). We obtain \( F(a, b; c; \frac{z}{b}) \)

\(^1\)Ibid.
which converges for $|Z| < 1$.

$sF_1$ satisfies

$$Z \left(1 - \frac{\omega}{Z} \right) \omega'' + \left( c - \frac{a + \frac{1}{2} + n}{\omega} \right) \omega' - \omega = 0$$

If we let $\omega \to \infty$, we obtain (formally) the series

$$sF_1(a; c; Z) = \frac{n(n-1)}{n(n-1)} \sum \frac{n(n-1)}{n(n-1)} \frac{Z^n}{n!}$$

which converges for $|Z| < 1$. Moreover, $sF_1(a; c; Z)$ satisfies

$$Z \omega'' + (c - Z) \omega' + a \omega$$

Now, if we replace $a$ by $-a$ and take $c = 1$ we have $sF_1(-a; 1; Z)$ which satisfies

$$Z \omega'' + (1 - Z) \omega' + a \omega$$

This is known as Laguerre's differential equation which has polynomial solutions $L_n(Z)$ defined as follows:

$$L_n(Z) = n! \cdot sF_1(-n; 1; Z)$$

Hence, we can write the following definition:

$$L_n(Z) = e^Z \frac{d^n}{dz^n} \left( Z^n e^{-Z} \right).$$

### Legendre Functions

Let us consider the hypergeometric equation defined as the function

$$F \left(a, a; r; Z\right)$$

Now, if we let $a = -n$ and $r = n + 1$, where $n$ is a positive integer, with

$r = 1$, then choose $Z = \frac{1 - t}{a}$, then we have $F (-n, n + 1; 1; \frac{1 - t}{a})$ as a solution of the differential equation:

$$(1 - t^2) \frac{d^2 \omega}{dt^2} - 2t \frac{d \omega}{dt} + n(n+1) \omega = 0$$

This equation is called Legendre's differential equation.
Since $F(-n, n+1; 1; \frac{1-x}{2})$ is a solution of Legendre's Equation and we write $P_n(x) = F(-n, n+1; 1; \frac{1-x}{2})$, where $P_n(x)$ is defined for all values of $n$ as the Legendre Function of degree $n$ of the First Kind.
CHAPTER V
CONCLUSION

Special functions of mathematical physics were defined as those functions which satisfy particular linear differential equations as a solution.

We have given an analysis of the second-order linear differential equation with variable coefficients.

Our object of investigation was to determine if this second-order linear differential equation had a solution, and if so, what effect the singularities of the variable coefficient have upon its solution.

In our discussion of solutions of differential equations by the method of Frobenius, we consider the properties of the indicial equation and how the nature of its roots determined the type of solution which would satisfy the second-order linear differential equation. If the variable coefficients possessed a regular singularity, then the differential equation possessed a Frobenius type series solution about the regular singularity.

These particular differential equations were considered from a complex-variable point of view to gain a greater insight into the nature of their solutions, especially those which possessed a finite number of singular points.
BIBLIOGRAPHY

Books


