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On the methods of constructing quaternions

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ON THE METHODS OF CONSTRUCTING QUATERNIONS

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INTRODUCTION

The Theory of Quaternions was constructed by Sir William Rowan Hamilton, a Royal Astronomer of Ireland, who presented his first paper on the subject to the Royal Irish Academy in 1843. His lectures on Quaternions were published in 1843, and his Elements in 1866. The second edition of his Elements was published in two volumes, with notes and appendices by C. J. Joly, London, 1899.

In 1895 the International Association for the Study of Quaternions was established, which published a bibliography of nearly 1,000 books and papers (1904) on the subject of Quaternions. This should have advanced the subject greatly; but the Association ended in 1913 with the death of the president of the Association.

Since that time many contributions and applications have been made, as is evident by the following bibliography:

L. Brand, "The Roots of Quaternions," American Mathematical Monthly, vol. 49 (1928), p. 519.

R. L. Carstens, "A Definition of Quaternions by Independent Postulates," American Mathematical Monthly, vol. 12 (1919), p. 396.

H. S. M. Coxeter, "Quaternions and Reflections," American Mathematical Monthly, vol. 9 (1943), p. 136.

L. E. Dickson, "A Matrix Defined by the Quaternion Group," American Mathematical Society, vol. 9 (1916), p. 243.

L. W. Griffiths, "Generalized Quaternion Algebras and Theory of Numbers," American Mathematical Journal, vol. 50 (1928), pp. 304-314.

D. E. Littlewood, "The Solution of Linear Congruences in Quaternions," American Mathematical Society's Bulletin, vol. 32 (1931), p. 343.

In this investigation we will give two methods of constructing

Quaternions. In Chapter I, we will give the geometrical construction of Quaternions using the definition of a right versor. In Chapter II we will give an algebraic construction of Quaternions.

CHAPTER I

ON A SYSTEM OF THREE RIGHT VERSORS, IN THREE RECTANGULAR PLANES, AND ON THE LAWS OF THE SYMBOLS, I, J, K.

A right versor is an Operator, which turns a line, in a plane perpendicular to itself, through a positive quadrant of rotation: and thereby to oblige the operand-Line to take a new direction, at right angles to its old direction, but without any change of length [1; 143].

Suppose that OI, OJ, OK are any three given and co-initial but rectangular Unit-lines, the rotation round the first to second to third being positive; and let OI', OJ', OK' be the three Unit-vectors respectively opposite to these, so that

$$OI' = OI; \; OJ' = -OJ; \; OK' = -OK.$$

Let the three symbols i, j, k denote a system of three right vesors, in three mutually rectangular planes, with the three given lines for their respective axes, so that the axis of i = OI, the axis of j = OJ, and the axis of k = OK and

$$i = OK:OJ, \quad j = OI:OK, \quad k = OJ:OI$$

as will be illustrated by the figures:

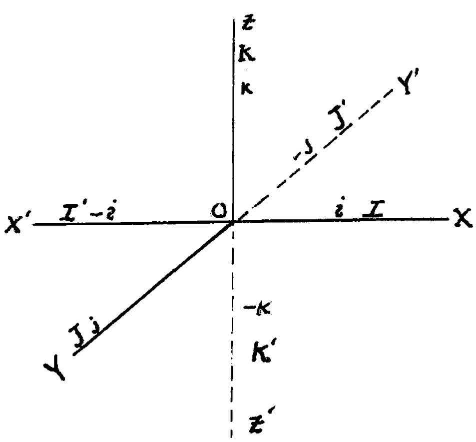

Fig. 1

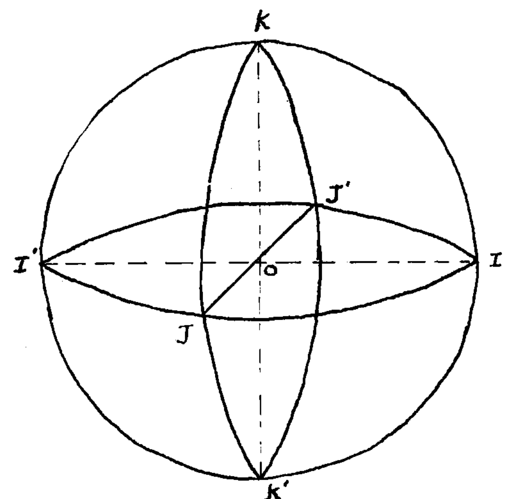

Fig. 2

We have then these other expressions from the same three versors:

$$
\begin{aligned}
i &= OJ':OK = OJ:OK' = OK':OJ' \\
j &= OK':OI = OK:OI' = OI':OK' \\
k &= OI':OJ = OI:OJ' = OJ':OI'
\end{aligned}
$$

and the three respectively opposite versors may be expressed:

$$
\begin{aligned}
-i &= OJ:OK = OK':OJ = OJ':OK' = OK:OJ' \\
-j &= OK:OI = OI':OK = OK':OI' = OI:OK' \\
-k &= OI:OJ = OJ':OI = OI':OJ' = OJ:OI'
\end{aligned}
$$

From the foregoing expressions several important symbolical consequences follow:

1. $i^2 = (OJ':OK) \,.\, (OK:OJ) = OJ':OJ$
2. $j^2 = (OK':OI) \,.\, (OI:OK) = OK':OK$
3. $k^2 = (OI':OJ) \,.\, (OJ:OI) = OI':OI$

Since $OI' = -OI$, $OJ' = -OJ$ and $OK' = -OK$.

4. We have the following values for the square of the symbols, i, j, and k:

$$i^2 = -1, \; j^2 = -1, \; k^2 = -1$$

Since $i.j = (OJ:OK') \cdot (OK':OI) = OJ:OI = k$

5. We have the following relations for the products of i, j, k taken two by two, and in a certain order of succession:

$$ij = k; \quad jk = i; \quad ki = j.$$

And since $j \cdot i = (OI:OK) \cdot (OK:OJ) = OI:OJ$, which is the opposite order of k and we have the following relations when the factors are taken in certain opposite order of succession:

$$ji = k \quad kj = -i; \quad ik = -j.$$

Figure 3 illustrates the foregoing multiplication:

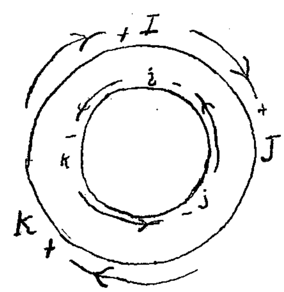

Fig. 3

To show the relationship between such multiplication of i, j, and k with the theory of representative arcs and angles, it will be necessary to cut figure 2. If we cut the sphere in figure 2, through the points K, J', K', J, we get a great circle passing through the poles of the sphere, or a meridian of the sphere.

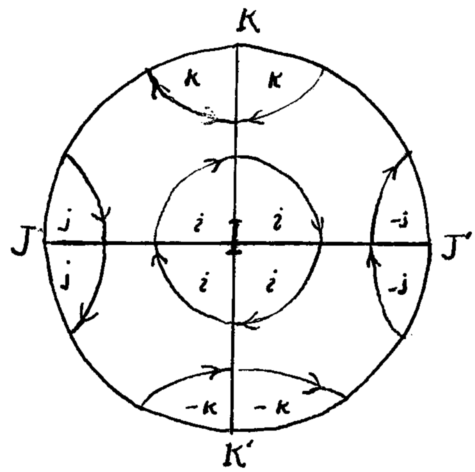

Fig. 4

We may regard one of the four quadrantal arcs, JK, KJ', J'K', K'J, or any of the four spherical right angles, JIK, KIJ', J'IK', K'IJ, where the arcs subtend at their common pole I, as representing the versor i; and similarly for j and k, with I' opposite to I, which is to be thought of as being in back of I [1; 155].

Thus the squaring of i amounts to adding the quadrant KJ' to an equal quadrant JK whose sum is the great semi-circle JJ' which represents negative unity, and a similar addition for j and k.

The multiplication of the $ij = k$, $jk = i$ and $ki = j$, and the contrasted multiplication of $ji = -k$, $ik = -j$ and $kj = -i$ may in a similar manner be arcually constructed.

We can readily see that the commutative property of multiplication does not hold, for $ij = k$ but $ji = -k$. However, the associative property holds:

$$ijk = -1 \text{ or } ij \cdot k = kk = k^2 = -1$$

and

$$kij = ki \cdot j = j = j^2 = -1$$

Therefore in general the associative law exists for any three symbols in any order.

We may therefore establish the following formulae

$$i^2 = j^2 = k^2 = ijk = -1 \qquad (I)$$

to which we shall occasionally refer, and to which, we shall find, contain virtually all the laws of the symbols ijk, and therefore to be a sufficient symbolical basis for the whole calculus of quaternions because it will be shown that every quaternion can be reduced to the <u>Quadrinomial Form</u>,

$$q = a_0 + a_1 i + a_2 j + a_3 k$$

where a_0, a_1, a_2, a_3 compose a system of four scalars, while i, j and k are the same three right versors discussed above.

Therefore the addition and subtraction of quaternions simply reduce these two operations to the addition and subtraction of coefficients as the following example will show.

Set $q = a_0 + a_1 i + a_2 j + a_3 k$

and

$$p = b_0 + b_1 i + b_2 j + b_3 k$$

then

$$q \pm p = a_0 \pm b_0 + i\ (a_1 \pm b_1) + j\ (a_2 \pm b_2) + k\ (a_3 \pm b_3).$$

To construct the product of two general quaternions q and p multiply each term of q by each term of p. Thus the product of two quaternions

$$q \cdot p = R(qp) + V(qp)$$

where R(qp) and V(qp) are the scalar and vector parts respectively of the product.

We have shown that the commutative law fails. However, it is easy to see that the distributive law of multiplication holds without change.

The quaternion $\overline{q} = R(q) - V(q)$ is called the conjugate of q and conversely q is said to be the conjugate of $\overline{q}$. Thus the pair of quaternions q and $\overline{q}$ are said to be mutually conjugate quaternions. The product of any two mutually conjugate quaternions q and $\overline{q}$ is defined as the norm of q.

Thus $q \cdot \overline{q} = a_0^2 + a_1^2 + a_2^2 + a_3^2 = N(q)$ where $N(q)$ is the symbolical representation of the norm of q.

Now suppose we let a_o, a_1, a_2, a_3 be real numbers, so that q is a real quaternion; if $q \neq 0$, then, $N(q) \neq 0$ and q has an inverse q^{-1} or $q^{-1} = \dfrac{\overline{q}}{N(q)}$.

If $q \neq 0$, then $qx = p$ has the unique solution $x = q^{-1}p$, and $xq = p$ has the unique solution $x = pq^{-1}$, so that both the right hand and the left hand division is always uniquely determined if the divisor is a real quaternion not equal to zero.

The conjugate of the product $q \cdot p$ is equal to the product of the conjugates $\overline{p} \cdot \overline{q}$ taken in the reverse order.

The norm of $p \cdot q$ or $N(qp)$ is $p.q\ (\overline{p.q}) = pq \cdot \overline{qp}$ by definition. By the associative law this may be written as

$$N(p.q) = p\ (q\ \overline{q})\overline{p} = p(N_{(q)})\overline{p}$$

since $N(q)$ is an ordinary number it is commutative with $\overline{p}$, therefore

$$N(pq) = p\ \overline{p}.N(q) = N(p)\ N(q).$$

and therefore the result we get can be stated as follows: the norm of the product of two quaterions is equal to the product of the norms.

CHAPTER II

THE ALGEBRAIC CONSTRUCTION OF QUATERNIONS

Consider the quadruples $(a_o, a_1, a_2, a_3,)$ of real or complex numbers.

Set $\alpha = (a_o, a_1, a_2, a_3,)$

Define $(a_o, a_1, a_2, a_3) = (b_o, b_1, b_2, b_3)$

to mean $a_o = b_o \quad a_1 = b_1, a_2 = b_2, a_3 = b_3$.

Define addition by the identity:

$$(a_o, a_1, a_2, a_3) + (b_o, b_1, b_2, b_3) = (a_o + b_o, a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

and scalar multiplication by the identity:

$$K(a_o, a_1, a_2, a_3) = (Ka_o, Ka_1, Ka_2, Ka_3) = (a_o, a_1, a_2, a_3)K.$$

It follows then that

$$\alpha = (a_o, a_1, a_2, a_3) = a_o (1,00,0) + a_1 (0,1,0,0) + a_2(0,0,1,0) + a_3(0,0,0,1).$$

Then

$$\alpha\beta = (a_o, a_1, a_2, a_3)(b_o, b_1, b_2, b_3) = (A,B,C,D)$$

Where $A = a_o b_o - a_1 b_1 - a_2 b_2 - a_3 b_3$;

$B = a_o b_1 + a_1 b_o + a_2 b_3 - a_3 b_2$;

$C = a_o b_2 + a_2 b_o + a_3 b_1 - a_1 b_3$; and

$D = a_o b_3 + a_3 b_o + a_1 b_2 - a_2 b_1$.

Therefore it follows that

$(1,0,0,0)^2 = (1,0,0,0)$, and

$(0,1,0,0)^2 = (0,0,1,0)^2 = (0,0,0,1)^2 = (-1,0,0,0)$,

Consider now the particular quadruples

$1 = (1,0,0,0)$; $i = (0,1,0,0)$; $j = (0,0,1,0)$; $K = (0,0,0,1)$,

then $i^2 = j^2 = k^2$; $ij = K$, $ji = -K$, $jK = i$, $Kj = -i$, $Ki = j$, $iK = -j$

therefore

$$(a_o, a_1, a_2, a_3) = a_o + a_1 i + a_2 j + a_3 K.$$

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